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SIMPLE MODELS OF THE ENERGETICS OF TRANSVERSELY VIBRATING PLATES

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Approximate energy models for infinite and finite plates are derived using three relationships; an energy balance, a loss factor damping model and an approximate energy transmission model. The energy transmission relationship for the infinite plate relates the far field radial intensity, the group speed and the local energy density. The resulting energy equation for infinite plates is an excellent approximation in the far field. For finite plates, the far field energy density and intensity expressions for plane wave approximations are smoothed in order to derive an energy transmission relationship. The resulting relationship is analogous to Fourier's law of heat conduction. The energy model is a second order equation which models the smoothed far field energy distribution. The equations model the general behavior of finite plates well and explain the dependence of plate energetics on frequency and damping.

1. INTRODUCTION

At high frequency, approximate methods for predicting the response of structures are often preferable to exact methods. At high frequency, where wavelengths are short, the response varies significantly as a function of location and frequency. The exact nature of this response is often not important and may disguise the general behavior of the structure [1]. At high frequency or for broadband analysis, exact solutions are also computationally intensive. For many diagnostic and design purposes, a smoothed approximation is often preferred.

Statistical Energy Analysis (SEA) was the first method developed as an approximate analytical method for built-up structures [2]. The method has been widely utilized [3–5]. However, SEA is a lumped parameter approach and, as such, it models the behavior of any subsystem as a single parameter. Thus, SEA is not capable of modelling local behavior. This characteristic makes model development difficult. The analyst is required to determine equivalent global characteristics for features which are local. Thus, considerable judgement is required of the analyst.

Consequently, there have been investigations of other methods to predict the approximate response of structures on a continuous basis. Several of these investigations used known energy flow relationships for ribbed structures to develop continuum models of energy which have been implemented using the finite element method [6, 7]. Nefske and Sung extended these methods to beams but did not explain the difference between the exact solution and their energy solution, which was implemented using the finite element method [8]. Wohlever and Bernhard developed a rigorous method for developing approximate energy solutions which was refined further by Bouthier [9, 10]. These models for rods, beams and membranes have been reported previously [9, 11]. In this paper, the method will be applied to transversely vibrating plates.

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An alternative exact energy solution has also been developed by Luzzato [12], Lase and Jezequel [13] and Le Bot and Jezequel [14]. This formulation uses the energy density and the Lagrangian energy as coupled variables. The resulting formulation is complete and, thus, can be used at any frequency without approximation. However, the computational requirements are at least as extensive as traditional analysis and, thus, no improvement in efficiency is achieved.

Plates constitute a major building block of many structures of interest to structural dynamicists and acousticians. Structures such as ships, aircraft, automobiles and buildings use plates as structural members. Plates often radiate a significant part of the acoustical energy of such systems. Therefore, it is desirable for engineers to have useful approximate methods for predicting energy flow in plates subject to high frequency vibrations as an alternative to classical methods of analysis.

The derivation of the expressions which govern the flow of energy in a transversely vibrating plate are discussed in this investigation. In section 2, the fundamental concepts of energy flux are discussed. In section 3, the motion of an infinite plate is expressed in terms of energy variables. In section 4, the motion of finite plates is expressed in terms of energy variables, and in section 5, several analytical verification studies are shown.

2. DERIVATION OF THE ENERGY EQUATIONS

The derivation of the energy governing equation uses three relationships: (1) a transmission equation relating energy density and intensity; (2) an energy balance equation from continuum mechanics; and (3) an energy loss relationship, where dissipated power is related to the local energy density in the vibrating medium using a loss factor. These three relationships are coupled to develop equations which govern the energy density in various vibrating systems.

The transmission relationship is unique to the structure of interest and is the point in the development of the energy equations at which significant approximations are typically made. The simplest energy equations result when the transmission behavior in the structure is analogous to Fourier's Law of heat conduction. Wohlever and Bernhard found that Fourier's Law behavior models energy flow in rods [9]. In beams, such behavior is only true for smoothed variations of energy and intensity. Bouthier and Bernhard showed that Fourier's Law behavior was also true for smoothed plane wave approximations of the energy and intensity in membranes [11]. The transmission relationship for infinite and finite plates will be derived later in this paper.

An energy balance in an elastic medium can be described using a control volume approach. The flow of energy across any given closed surface is equivalent to the rate of change of the total energy inside the surface which encloses the volume:

$$\iiint (\partial e/\partial t)\,\mathrm{d}V = \iint (\sigma \cdot (\partial \vec{u}/\partial t)) \cdot \mathrm{d}A + \iiint (\pi_{in} - \pi_{diss})\,\mathrm{d}V, \tag{1}$$

where e is the energy density inside the control volume, $\vec{u}$ is the displacement vector of any particle on the boundary of the control volume, π_{in} is the input power density (or energy input per unit volume per unit time), π_{diss} is the power density dissipated (or energy per unit volume dissipated per unit time), dA is the vector normal to the surface of the control volume for a given point on the surface, and

$$\sigma = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}. \tag{2}$$

In classical elasticity literature, σ is commonly called the stress dyadic. The change of energy density inside the control volume is due to the work done by the stresses on the medium. In terms of stress and velocity, the local outflow of energy density from the control volume is

$$\vec{I} = -\sigma \cdot (\partial \vec{u}/\partial t). \tag{3}$$

The units of $\vec{I}$ are power per unit area, which is intensity.

As discussed by Noiseux [15], and reviewed in detail by Bouthier [16], the exact expression for intensity in plates can be integrated across the thickness of the plate, and expressed in terms of applied loads and motion responses as

$$\bar{I}_x = -M_{xx}\frac{\partial^2 w}{\partial x\,\partial t} - M_{xy}\frac{\partial^2 w}{\partial y\,\partial t} + Q_x\frac{\partial w}{\partial t}, \qquad \bar{I}_y = -M_{yy}\frac{\partial^2 w}{\partial y\,\partial t} - M_{yx}\frac{\partial^2 w}{\partial x\,\partial t} + Q_y\frac{\partial w}{\partial t}, \tag{4, 5}$$

where overbars are used in this paper to designate variables which have been integrated across the thickness of the plate (e.g. $\bar{\vec{I}}$ is the integrated intensity, power per unit length), and

$$M_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right), \qquad M_{yy} = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right), \tag{6, 7}$$

$$M_{xy} = -D(1-v)\frac{\partial^2 w}{\partial x\,\partial y}, \tag{8}$$

and

$$M_{xy} = M_{yx}, \tag{9}$$

where M_{xx}, M_{xy} and M_{yy} are bending moments applied to the plate, D is the bending stiffness, v is the Poisson ratio, w is the transverse displacement, and Q_x and Q_y are shear forces. The intensity components in equations (4) and (5) have units of power per unit length.

By using the divergence theorem, the energy flux in equation (1) can be rewritten as

$$\iint_A (\sigma \cdot (\partial \vec{u}/\partial t)) \cdot \mathrm{d}A = \iint_A \vec{I} \cdot \mathrm{d}A = \iiint_V \nabla \cdot \vec{I}\,\mathrm{d}V. \tag{10}$$

Thus, equation (1) becomes

$$\iiint_V (\partial e/\partial t)\,\mathrm{d}V = \iiint_V (\pi_{in} - \pi_{diss} - \nabla \cdot \vec{I})\,\mathrm{d}V, \tag{11}$$

or

$$\partial e/\partial t = \pi_{in} - \pi_{diss} - \nabla \cdot \vec{I}. \tag{12}$$

Equation (12) is an energy balance relationship for all elastic media and is valid for steady or transient analysis. For the analysis of steady state vibrational energy propagation, the time variation of energy is zero and equation (12) becomes

$$\pi_{in} = \pi_{diss} + \nabla \cdot \vec{I}. \tag{13}$$

To account for losses in the medium, an expression for the loss of energy is needed. Using a hysteretic damping model, Cremer and Heckl [17] showed that the energy dissipated in one period at a point in an elastic medium vibrating harmonically in time with circular frequency ω is

$$e_{diss} = 2\pi\eta\langle e\rangle. \tag{14}$$

Since the period of oscillation is $\tau = 2\pi/\omega$, the time averaged dissipated power, π_{diss}, is

$$\langle \pi_{diss} \rangle = e_{diss}/\tau = \eta\omega\langle e \rangle. \tag{15}$$

Equation (15) is an energy loss relationship which has been derived using the hysteresis damping model. It assumes that kinetic and potential energy are approximately equal and $\eta \ll 1$.

3. ENERGY GOVERNING EQUATIONS FOR INFINITE PLATES

The equation of motion for a thin, transversely vibrating plate excited at a point is

$$D(1 + i\eta)\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = F\delta(r)\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{16}$$

where ρ is the density, h is the thickness, and η is the hysteresis damping coefficient. For an infinite plate excited by a point force, the solution will be uniform in the circumferential direction. The general solution of equation (16) is a linear combination of Hankel functions

$$w = \frac{F\omega}{8Dk^2}\,(\mathrm{H}_0^{(2)}(kr) - \mathrm{H}_0^{(2)}(-\mathrm{i}kr))\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{17}$$

where k is the wavenumber:

$$k = (\omega^2(\rho h/D))^{1/4}. \tag{18}$$

For this investigation of infinite plates, the farfield asymptotic expansion of equation (17) is used to obtain the farfield solution [18]

$$w = \frac{\mathrm{i}F}{8Dk^2}\left(\frac{2}{k\pi r}\right)^{1/2} \mathrm{e}^{\mathrm{i}(\omega t - kr + \pi/4)}. \tag{19}$$

As a rule of thumb, if a characteristic wavelength is denoted as λ, the far field is considered to be $\lambda/2$ away from any boundary, discontinuity or excitation [15]. The only term in equation (19) which contains propagation information is the exponential term. The propagation of energy of a cylindrical wave, such as that represented by equation (19), is in the radial direction. Miklowitz [19] shows that for the far field response of the form shown in equation (19), the relationship between energy and radial intensity using equations (4) and (5) is

$$I_r = c_g e, \tag{20}$$

where c_g is the group speed, which for plates is

$$c_g = 2(\omega^2(D/\rho h))^{1/4}. \tag{21}$$

Equation (20) is the energy transmission relationship for infinite plates and is valid for steady state or transient analysis. However, equation (20) only holds for one-dimensional (radial) analysis of wave propagation in the farfield of infinite systems.

By combining the far field energy transmission equation (20), the energy balance equation (13) and the energy dissipation equation (15), the governing differential equation for energy in an infinite plate excited at a point is

$$\frac{c_g}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\langle e \rangle) + \eta\omega(\langle e \rangle) = \langle \pi_{in} \rangle. \tag{22}$$

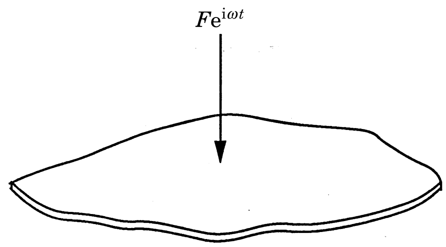

Figure 1. An infinite plate excited by a time harmonic point force.

The Green's function of equation (22) is

$$\langle e \rangle = \frac{A}{r}\, e^{-\eta\omega r/c_g}, \tag{23}$$

where A is solved by using input power information. The power input must balance the total power in the radial direction at some distance r_0 close to the excitation. Using equations (20) yields

$$\langle \bar{\pi}_{in} \rangle = \lim_{r_0 \to 0} 2\pi r_0 \bar{e} c_g. \tag{24}$$

Thus, the constant A is

$$A = \langle \bar{\pi}_{in} \rangle / 2\pi c_g. \tag{25}$$

The energy density solution from equations (22) and (25) is

$$\langle \bar{e} \rangle = \frac{\langle \bar{\pi}_{in} \rangle}{2\pi r c_g}\, e^{-\eta\omega r/c_g}. \tag{26}$$

This energy density distribution is identical to the energy density predicted by the far field displacement solution and, thus, no verification results will be shown here. However, at this point some analytical results will be shown to demonstrate the importance of damping in equation (26). The example presented here is for an infinite plate excited by a point force, as shown in Figure 1. The energy density in the plate is shown in Figure 2 as a function of kr for damped and undamped plates. The plate is made of aluminum and is 1·0 mm

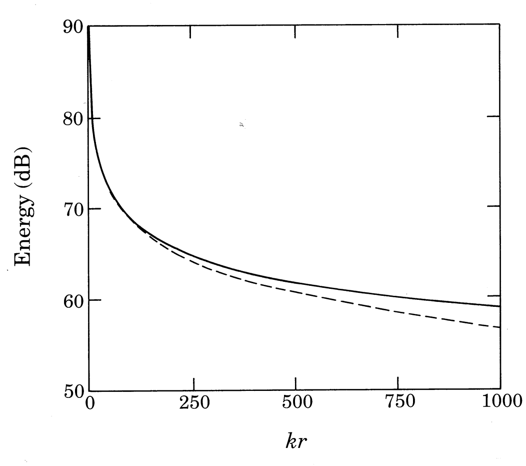

Figure 2. The energy density in the infinite plate: ——, undamped plate; – – –, damped plate. The reference energy density is 1×10^{-12} J/m^2.

thick. The magnitude of the force is 1·0 N and the damping is $\eta = 0{\cdot}001$. The effects of damping become apparent for large values of kr.

The energy equations work well as an approximate model for an infinite plate excited by a point force. For the analysis of finite systems, additional assumptions will be required because of complications brought about by the reflection of waves from the boundaries.

4. ENERGY EQUATIONS FOR FINITE PLATES

The reflection of waves at the boundaries of a finite plate greatly increases the complexity of the problem as illustrated by the analysis of energy flow in membranes [11]. The procedure used for the derivation of an energy governing equation in finite plates is essentially the same as the procedure followed for infinite plates. The flow of energy is due to energy dissipation at the boundary or within the medium because of damping.

For harmonic analysis of the homogeneous problem with $\eta \ll 1$, equation (16) can be rewritten as

$$\nabla^4 W - \frac{\rho h}{D}\omega^2(1 - \mathrm{i}\eta)W = 0, \tag{27}$$

where W is the complex amplitude of motion. The biharmonic operator in equation (27) is not separable in two dimensions. The solution of vibrating plates subject to various boundary conditions is available in the literature [20–22], but a complete, general solution of equation (27) is not available in closed form.

However, equation (27) can be factored into the form

$$(\nabla^2 + k^2)\cdot(\nabla^2 - k^2)W = 0, \tag{28}$$

where k is the wavenumber. To simplify the analysis and model the approximate response of plates, investigators frequently utilize the far field solution of equation (27) for the purpose of analyzing the flux of energy in plates [15]. The travelling plane wave component of the far field solution is

$$W_{ff} = (A_x\,\mathrm{e}^{-\mathrm{i}k_x x} + B_x\,\mathrm{e}^{\mathrm{i}k_x x})(A_y\,\mathrm{e}^{-\mathrm{i}k_y y} + B_y\,\mathrm{e}^{\mathrm{i}k_y y})\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{29}$$

The components of the wavenumber, k, are

$$k_x = k_{x1}(1 - \mathrm{i}\eta/4), \qquad k_y = k_{y1}(1 - \mathrm{i}\eta/4), \tag{30}$$

where

$$(k_{x1}^2 + k_{y1}^2)^2 = \rho h\omega^2/D. \tag{31}$$

The far field solution only satisfies the left side of the factored displacement equation (28). Thus, W_{ff} is not the complete solution of the displacement equation (16). The relationship between energy density and intensity variables will be established here in the far field of the vibrating plate using the far field solution.

The energy density in a vibrating plate is the sum of the kinetic and potential energy densities. The time averaged energy density is [23]

$$\langle \bar{e} \rangle = \frac{D}{4}\left(\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}\right)^* + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2}\right)^* + 2\nu\,\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2}\right)^*\right.$$
$$\left. + 2\nu(1-\nu)\,\frac{\partial^2 w}{\partial x\,\partial y}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^* + \frac{\rho h}{D}\frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial t}\right)^*\right). \tag{32}$$

The far field energy density can be found by substituting equation (29) into equation (32). The intensity in the plate is found by substituting equation (29) into the expressions for intensity shown in equations (4) and (5). The expanded expressions for energy and intensity in terms of the constants $A_x, \ldots, B_y$ are shown in Appendix C of Bouthier [16]. The constants would be determined by specifying the boundary conditions. However, the objective of this investigation is to find general relationships applicable for any type of boundary condition. Thus, the constants are retained and the relationships are derived to be true regardless of the values of the constants. Bouthier separated the expressions for energy density and intensity into four groups of four terms each. The expressions show no apparent relationship between energy density and intensity. When the intensity and energy density are smoothed by performing the operation

$$\langle \bar{e} \rangle = \frac{k_x k_y}{\pi^2} \int_{-\pi/2k_x}^{+\pi/2k_x} \int_{-\pi/2k_y}^{+\pi/2k_y} \langle e \rangle \, \mathrm{d}y \, \mathrm{d}x \tag{33}$$

at all locations on the plate, the smoothed energy density is

$$\begin{aligned} \langle \bar{e} \rangle = {} & (D/4)(|k_x|^2 + |k_y|^2) \\ & \cdot \{|A_x|^2|A_y|^2 \, \mathrm{e}^{-\eta(k_{x1}x + k_{y1}y)} + |A_x|^2|B_y|^2 \, \mathrm{e}^{-\eta(k_{x1}x - k_{y1}y)} \\ & + |B_x|^2|A_y|^2 \, \mathrm{e}^{\eta(k_{x1}x - k_{y1}y)} + |B_x|^2|B_y|^2 \, \mathrm{e}^{\eta(k_{x1}x + k_{y1}y)}\}, \end{aligned} \tag{34}$$

and the intensity components are

$$\begin{aligned} \langle \bar{I}_x \rangle = {} & k_x(\omega/2)(k_x^2 + k_y^2 + |k_x|^2 + \nu k_y^2 + (1-\nu)|k_y|^2) \\ & \cdot \{|A_x|^2|A_y|^2 \, \mathrm{e}^{-\eta(k_{x1}x + k_{y1}y)} + |A_x|^2|B_y|^2 \, \mathrm{e}^{-\eta(k_{x1}x - k_{y1}y)} \\ & - |B_x|^2|A_y|^2 \, \mathrm{e}^{\eta(k_{x1}x - k_{y1}y)} - |B_x|^2|B_y|^2 \, \mathrm{e}^{\eta(k_{x1}x + k_{y1}y)}\} \end{aligned} \tag{35}$$

and

$$\begin{aligned} \langle \bar{I}_y \rangle = {} & k_y(\omega/2)(k_y^2 + k_x^2 + |k_y|^2 + \nu k_x^2 + (1-\nu)|k_x|^2) \\ & \cdot \{|A_x|^2|A_y|^2 \, \mathrm{e}^{-\eta(k_{x1}x + k_{y1}y)} - |A_x|^2|B_y|^2 \, \mathrm{e}^{-\eta(k_{x1}x - k_{y1}y)} \\ & + |B_x|^2|A_y|^2 \, \mathrm{e}^{\eta(k_{x1}x - k_{y1}y)} - |B_x|^2|B_y|^2 \, \mathrm{e}^{\eta(k_{x1}x + k_{y1}y)}\}. \end{aligned} \tag{36}$$

The relationship between the smoothed variables $\langle \bar{e} \rangle$ and $\langle \vec{\bar{I}} \rangle$ is

$$\langle \vec{\bar{I}} \rangle = (c_g^2/\eta\omega)(\vec{i} \, \partial \langle \bar{e} \rangle / \partial x + \vec{j} \, \partial \langle \bar{e} \rangle / \partial y). \tag{37}$$

Equation (37) is the expression for the transmission of energy in a finite vibrating plate subject to losses.

Combining the energy balance equation (13), the power dissipation equation (15) and the transmission equation (37) yields the governing equation for the smoothed energy density in plates

$$(c_g^2/\eta\omega)(\partial^2 \langle \bar{e} \rangle / \partial x^2 + \partial^2 \langle \bar{e} \rangle / \partial y^2) + \eta\omega \langle \bar{e} \rangle = \langle \bar{\pi}_{in} \rangle. \tag{38}$$

Because of the solution form assumed in equation (29), equation (38) is derived for a plane wave approximation of the transverse vibration in the far field region of finite plates.

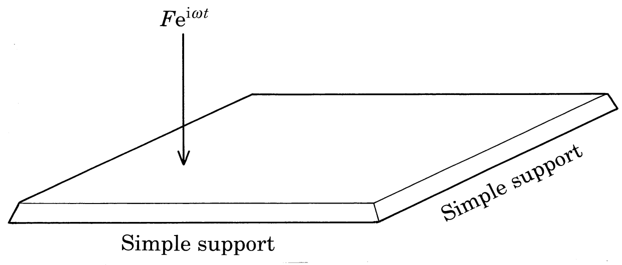

Figure 3. The plate response problem, displacement solution.

5. VERIFICATION

The accuracy of the energy density and the intensity field predicted by equations (37) and (38) is verified in this section. The results are compared to classical energy computed from modal analysis solutions for displacement. For each displacement solution case study, the plate was simply supported along its edges and excited by a pressure excitation at a single frequency over a small area at the center of the plate, as shown in Figure 3. The corresponding energy case studies use a distributed power input and zero energy flux (intensity) boundary conditions at the edges of the plate as shown in Figure 4. A distributed pressure excitation and power input were chosen instead of a point excitation because the two-dimensional Fourier series approximation converges faster and with fewer terms for distributed excitation. The excitation is located at the center of the plate and acts over a square patch 0·1 m × 0·1 m in size. The verification procedure was performed for pure tone excitations because the computational effort required to compute broadband results is so extensive. For the simulation performed in this section, the modal summation was truncated at 4900 modes to guarantee convergence of all the variables involved. The parameters of the plate are as follows: $E = 7{\cdot}1 \times 10^{10}$ N/m^2, $\rho = 2700$ kg/m^3, $h = 0{\cdot}001$ m, $F = 0{\cdot}01$ N, $l_x = l_y = 1{\cdot}00$ m.

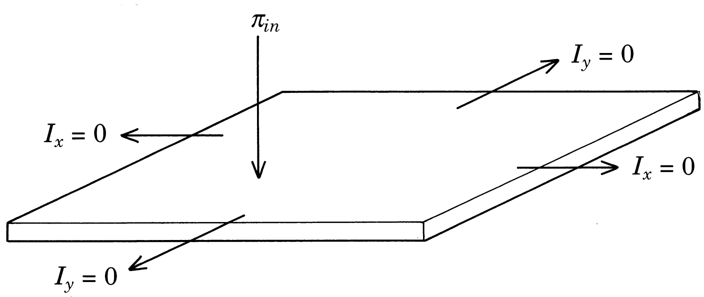

Figure 4. The plate response problem, energy solution.

TABLE 1

Frequency and loss factor for each case study

Case	Frequency (Hz)	Loss factor
1	239	0·05
2	239	0·20
3	487	0·05
4	487	0·20

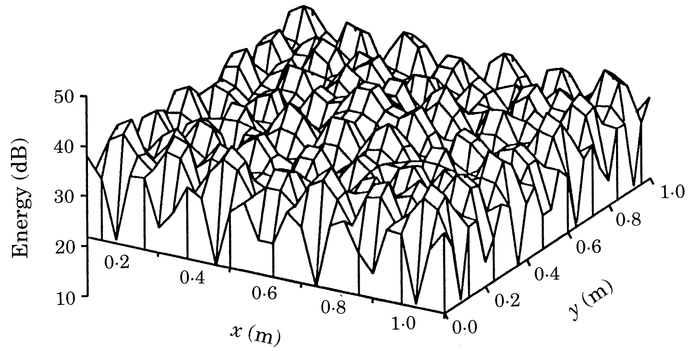

Figure 5. The analytical energy density distribution in the plate when $f = 239$ Hz and $\eta = 0{\cdot}05$. The reference energy density is 1×10^{-12} J/m^2.

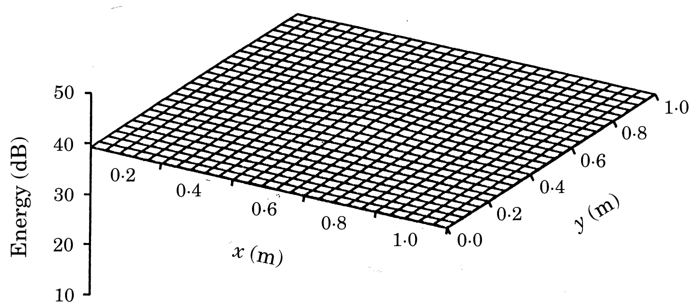

Figure 6. The approximate energy density distribution in the plate when $f = 239$ Hz and $\eta = 0{\cdot}05$. The reference energy density is 1×10^{-12} J/m^2.

For the verification procedure in this investigation, four different simulations were performed to test the effects of frequency and damping on the accuracy of the approximate energy density prediction. The frequency and damping for each case are summarized in Table 1. The input power used for the energy model was obtained from the modal solution by calculating the power of the source.

In the first example shown here (case 1), the frequency and damping are $f = 239$ Hz and $\eta = 0{\cdot}05$. The "exact" and approximate energy distributions are shown in Figures 5 and 6, respectively. A comparison of the energy density distributions along the line $x = y$ is shown in Figure 7. The classical solution deviates significantly from the approximate solution. However, the approximate solution is a good smoothed representation. The energy flow in the plate obtained from the classical and approximate solutions are shown in Figures 8

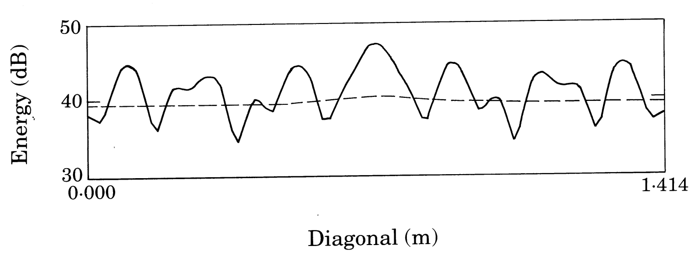

Figure 7. The energy density distribution comparison along the line $x = y$ when $f = 239$ Hz and $\eta = 0{\cdot}05$: ——, exact solution; – – – –, approximate solution. The reference density is 1×10^{-12} J/m^2.

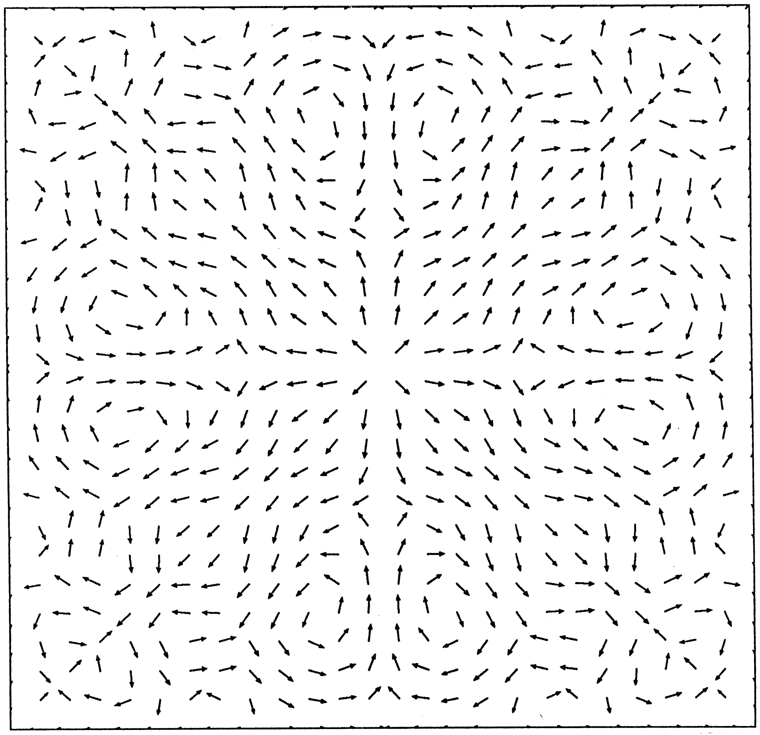

Figure 8. The analytical intensity in the plate when $f = 239$ Hz and $\eta = 0{\cdot}05$. The reference intensity is 1×10^{-12} W/m².

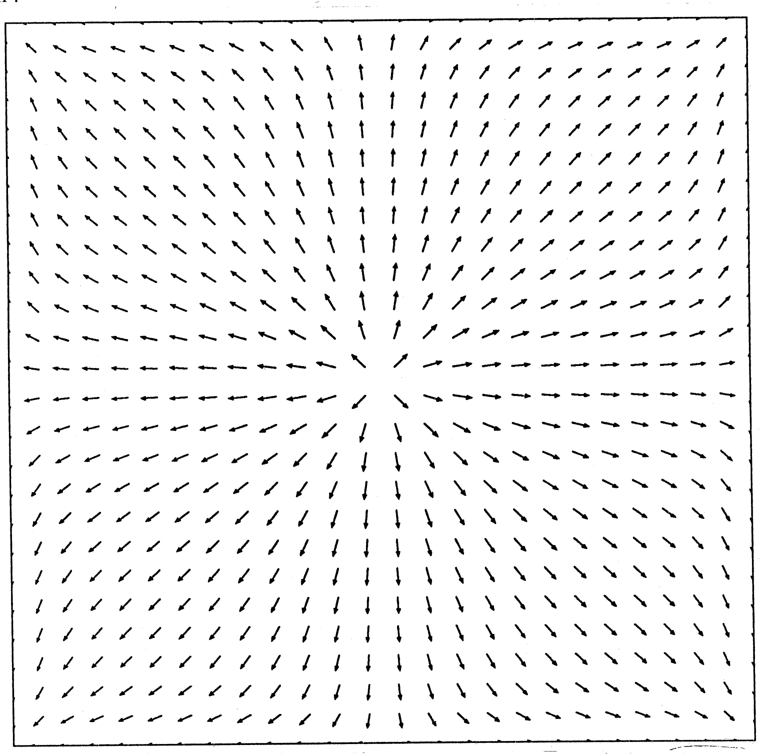

Figure 9. The approximate intensity in the plate when $f = 239$ Hz and $\eta = 0{\cdot}05$. The reference intensity is 1×10^{-12} W/m².

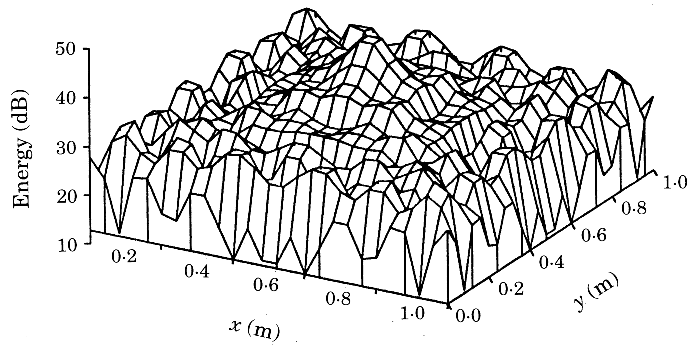

Figure 10. The analytical energy density distribution in the plate when $f = 239$ Hz and $\eta = 0{\cdot}20$. The reference energy density is 1×10^{-12} J/m^2.

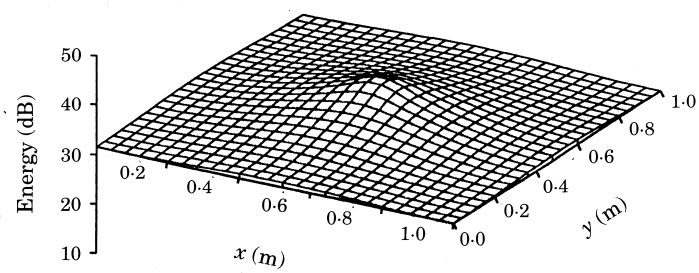

Figure 11. The approximate energy density distribution in the plate when $f = 239$ Hz and $\eta = 0{\cdot}20$. The reference energy density is 1×10^{-12} J/m^2.

and 9, respectively. Note that no energy flows across the boundary in either solution. The approximate solution for intensity is a good smooth representation of the "exact" intensity prediction.

As a second example (case 2), the damping is changed to $\eta = 0{\cdot}20$. The energy density distributions computed from modal analysis and from the solution of equation (38) are shown in Figures 10 and 11, respectively. A comparison of the energy density distributions along the line $x = y$ is shown in Figure 12. The intensity computed from modal analysis is shown in Figure 13 and the intensity computed from equation (37) is shown in Figure 14.

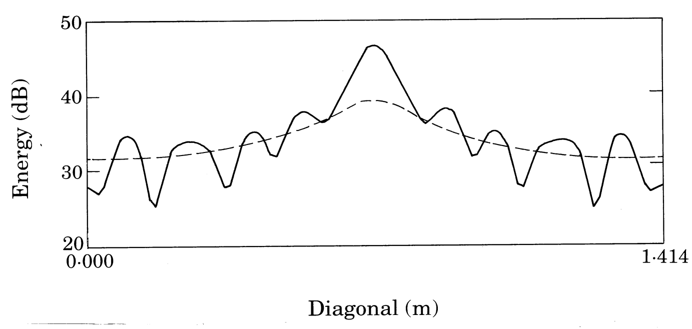

Figure 12. The energy density distribution comparison along the line $x = y$ when $f = 239$ Hz and $\eta = 0{\cdot}20$: ——, exact solution; – – – –, approximate solution. The reference energy density is 1×10^{-12} J/m^2.

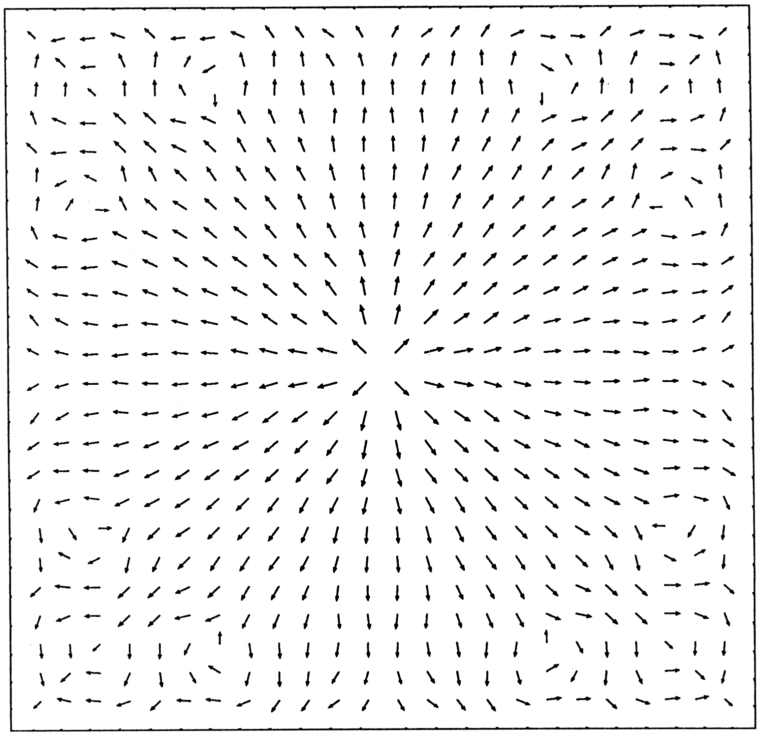

Figure 13. The analytical intensity in the plate when $f = 239$ Hz and $\eta = 0{\cdot}20$. The reference intensity is 1×10^{-12} W/m^2.

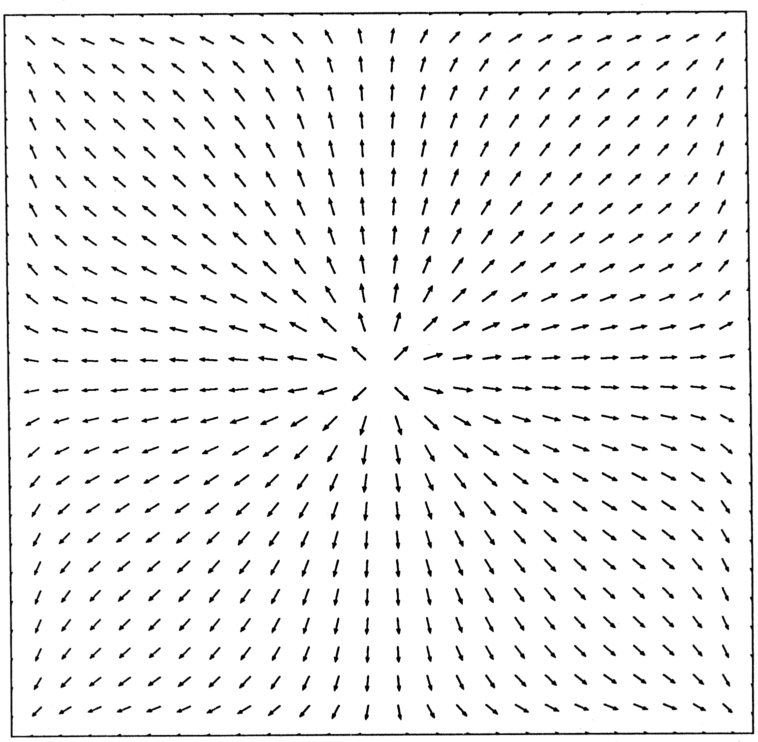

Figure 14. The approximate intensity in the plate when $f = 239$ Hz and $\eta = 0{\cdot}20$. The reference intensity is 1×10^{-12} W/m^2.

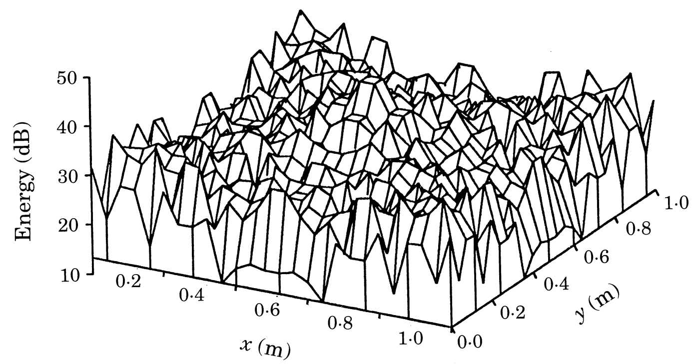

Figure 15. The analytical energy density distribution in the plate when $f = 487$ Hz and $\eta = 0{\cdot}05$. The reference energy density is 1×10^{-12} J/m^2.

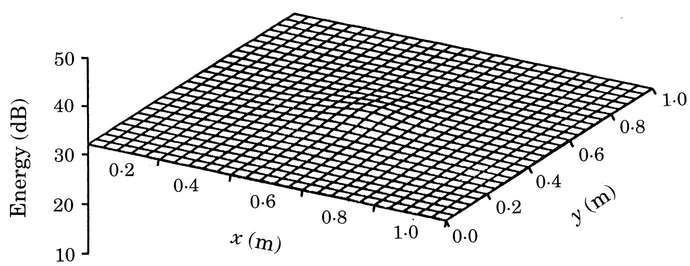

Figure 16. The approximate energy density distribution in the plate when $f = 487$ Hz and $\eta = 0{\cdot}05$. The reference energy density is 1×10^{-12} J/m^2.

In the first two examples ($f = 239$ Hz), the approximate solution improves in comparison to the classical solution as the damping is increased. When the damping is low, there is little global variation in the energy density, as shown in Figure 6. If the damping is increased at the same frequency, the global variation of the energy density increases, as shown in Figure 11. Furthermore, the energy flow becomes smoother as the damping is increased (see Figures 8 and 13).

For the next two simulations (cases 3 and 4), the frequency of excitation is $f = 487$ Hz. For case 3, the damping is $\eta = 0{\cdot}05$. The corresponding energy density distributions are shown in Figures 15 and 16. The energy density distribution along the line $x = y$ is shown in Figure 17. The intensity is shown in Figures 18 and 19. For the last example (case 4),

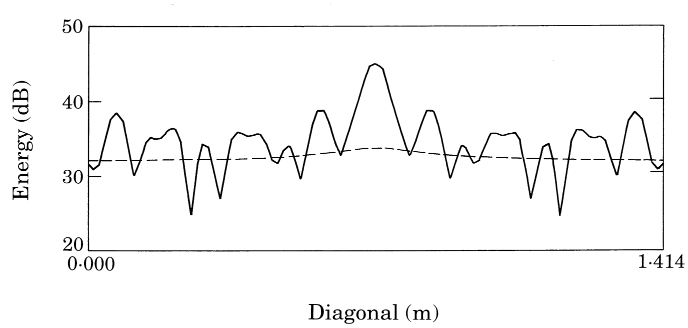

Figure 17. The energy density distribution along the line $x = y$ when $f = 487$ Hz and $\eta = 0{\cdot}05$: ——, exact solution; ––––, approximate solution. The reference energy density is 1×10^{-12} J/m^2.

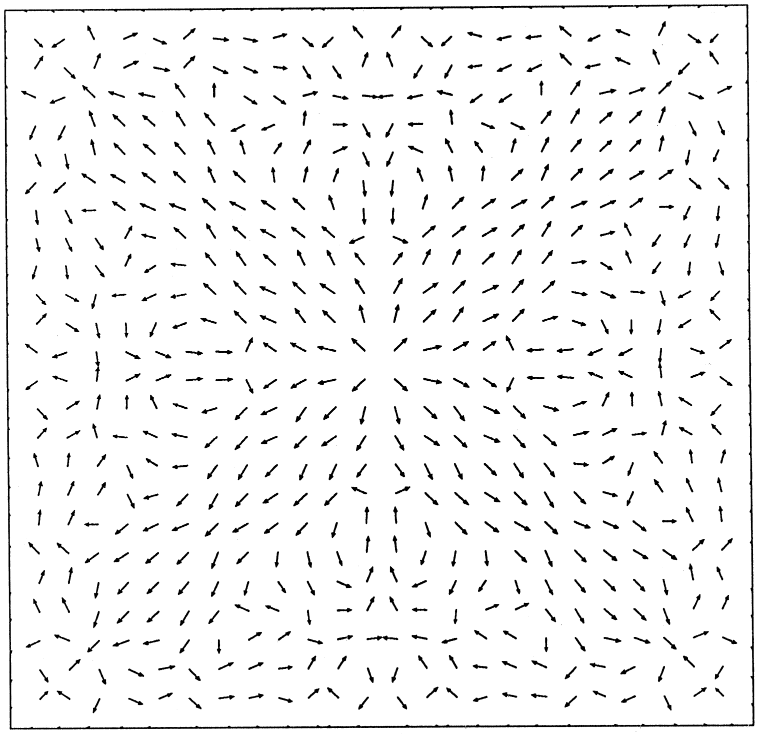

Figure 18. The analytical intensity in the plate when $f = 487$ Hz and $\eta = 0{\cdot}05$. The reference intensity is 1×10^{-12} W/m^2.

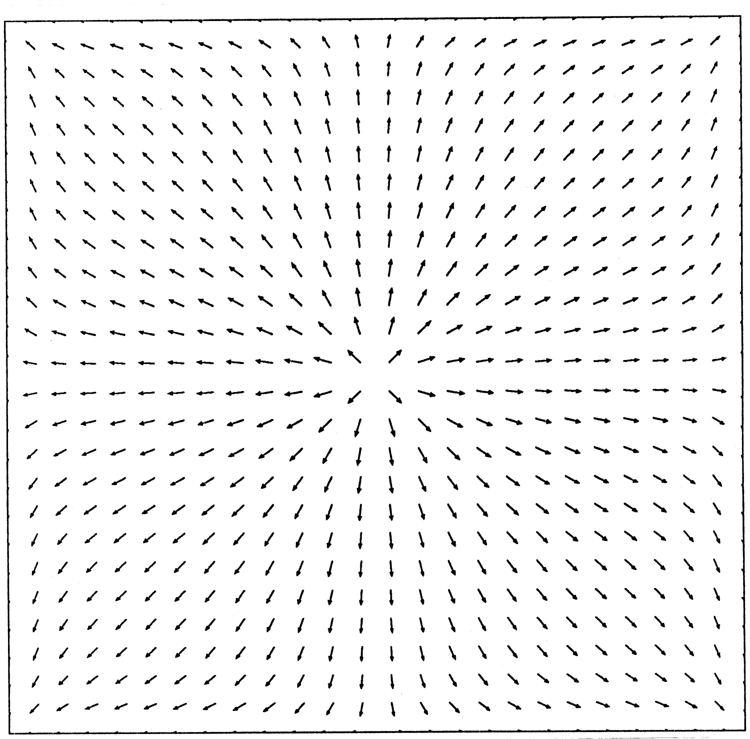

Figure 19. The approximate intensity in the plate when $f = 487$ Hz and $\eta = 0{\cdot}05$. The reference intensity is 1×10^{-12} W/m^2.

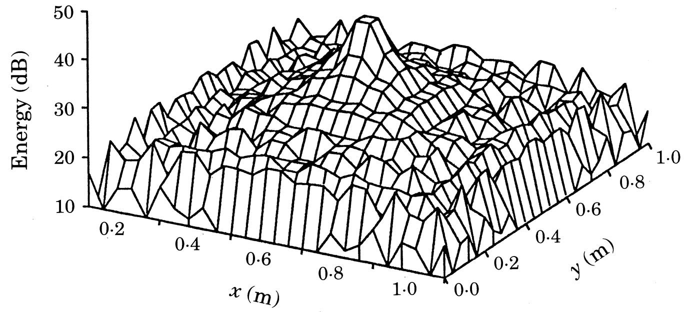

Figure 20. The analytical energy density distribution in the plate when $f = 487$ Hz and $\eta = 0{\cdot}20$. The reference energy density is 1×10^{-12} J/m^2.

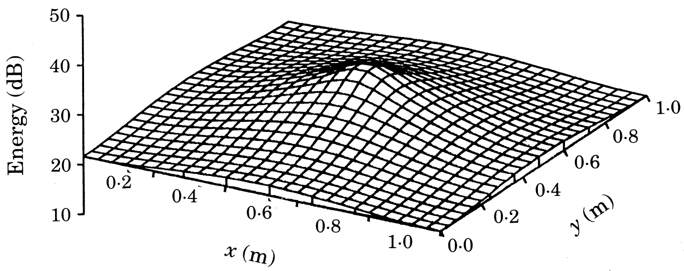

Figure 21. The approximate energy density distribution in the plate when $f = 487$ Hz and $\eta = 0{\cdot}20$. The reference energy density is 1×10^{-12} J/m^2.

the damping is $\eta = 0{\cdot}20$. The analytical and approximate energy density distributions are shown in Figures 20 and 21. The energy density distribution along the line $x = y$ is shown in Figure 22. The analytical and approximate intensity are shown in Figures 23 and 24.

For cases 3 and 4 ($f = 487$ Hz), the local energy density variation of the classical solution decreases and the global energy density variation increases as damping increases. Furthermore, the local variation of energy density decreases as the frequency of excitation increases for a fixed value of damping. The global energy density variation increases as

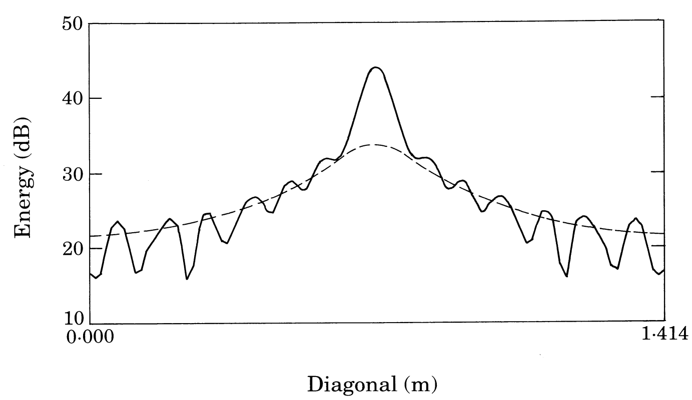

Figure 22. The energy density distribution comparison along the line $x = y$ when $f = 487$ Hz and $\eta = 0{\cdot}20$: ——, exact solution; – – –, approximate solution. The reference energy density is 1×10^{-12} J/m^2.

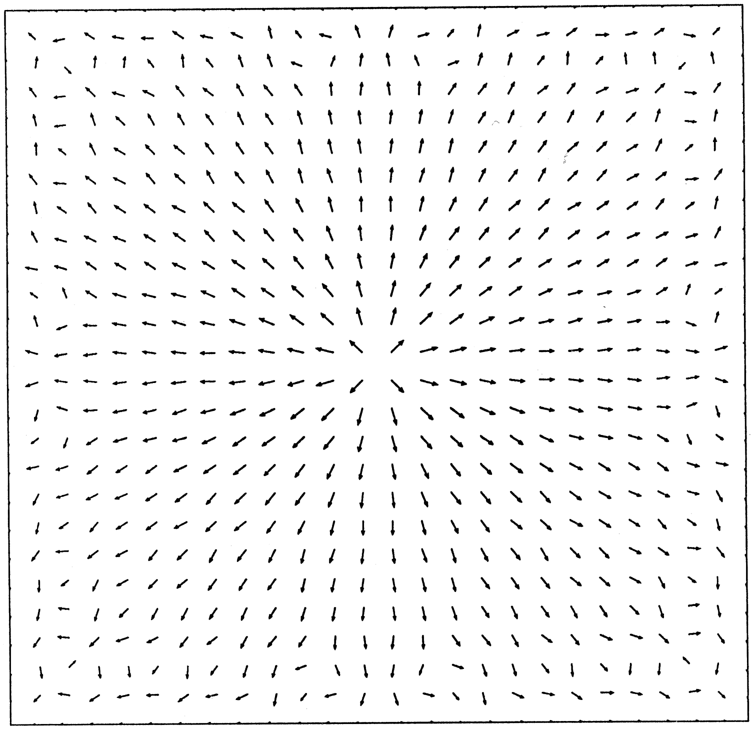

Figure 23. The analytical intensity in the plate when $f = 487$ Hz and $\eta = 0{\cdot}20$. The reference intensity is 1×10^{-12} W/m^2.

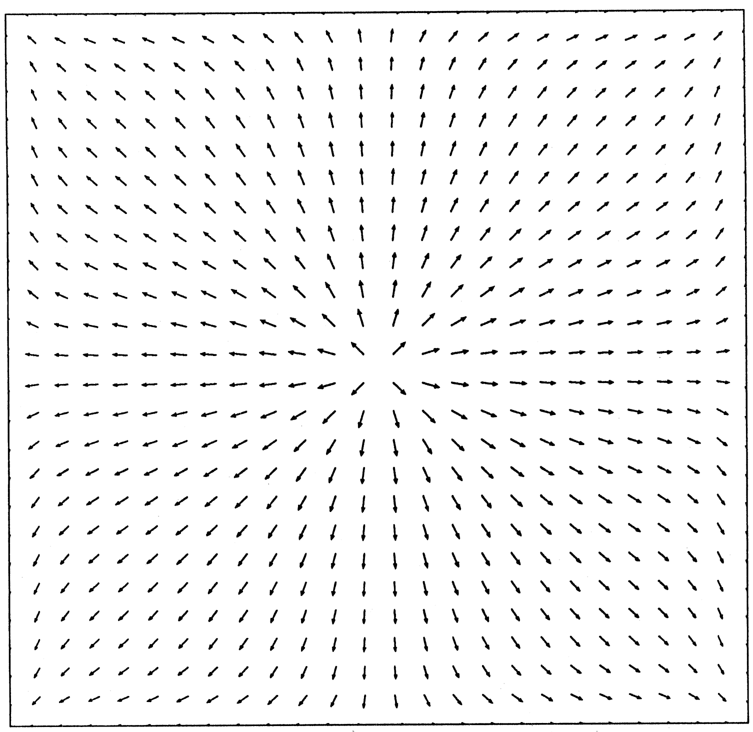

Figure 24. The approximate intensity in the plate when $f = 487$ Hz and $\eta = 0{\cdot}20$. The reference intensity is 1×10^{-12} W/m^2.

the frequency of excitation increases. These phenomena can be observed by comparing Figures 7 and 17 or Figures 12 and 22.

All four finite plate case studies are single frequency results. The energetics of finite plates for broadband excitation are expected to be spatially smoother than for the results shown in these case studies, as shown for membranes by Bouthier and Bernhard [11].

6. CONCLUSIONS

In this paper, approximate equations have been derived to model the energetics of transversely vibrating plates. For this development, a far field approximation has been used for steady state analysis with harmonic excitation. Three relationships were used to derive the energy equations. The energy loss relationship and the energy balance are straight-forward and commonly used. The energy transmission relationships are more difficult to derive. In this investigation, only far field approximations were discussed. For infinite plates, this approximation is sufficient to develop a first order energy equation which accurately predicts the far field energy density and intensity. For finite plates, the transmission relationship was found using a smoothed approximation of the far field, plane wave solution. The resulting second order energy equation retains the general character of the exact energy solution. It can be used to predict that energy decays as it propagates away from the source, and that the effect of damping as the loss factor or frequency increases.

These models will be relatively straightforward to implement using numerical methods such as the finite element method. The predictions are expected to be useful for understanding the general behavior of plate structures at high frequency.

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