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THEORY RELATING TO THE NOISE OF ROTATING MACHINERY

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This paper discusses and extends the theory of sound generation by multibladed single-stage fans operating in a free field. The results would also be applicable to shrouded fans provided that the shroud dimensions are small compared with the acoustic wavelength. The main point advanced is that it is inappropriate to regard the sound generation question as a boundary value problem governed by the homogeneous wave equation. Inhomogeneities of the equation caused by a finite velocity field in the vicinity of the fan induce a quadrupole distribution whose effect is studied. It is concluded that this effect is negligible only for low-speed, few-bladed fans in the first harmonics. For multibladed high-speed fans the quadrupole effects are important; both through the potential and turbulent velocity fields. In the absence of turbulence the inhomogeneous potential field may generate more sound than does the rotation of the steady blade loads, whereas the presence of turbulence provides a mechanism by which the potential field around the fan is scattered as sound. The theory of this mechanism is developed, and it becomes evident that it is vastly more important in generating the blade passage frequency sound heard near the axis of single-stage fans than any mechanism so far suggested. The paper then goes on to develop the theory of fan noise radiation at frequencies not necessarily related to the blade passage frequency, and is concluded by a formula relating the power spectral density of the fluctuating forces on a rotating blade to the spectral description of the radiation field.

1. INTRODUCTION

The earliest theoretical study of the noise generated by rotating machinery was the work of Gutin [1], who analysed the sound produced by a two-bladed aeroplane propeller. He discovered that the forces exerted by the propeller on the surrounding air generate the sound; they are equivalent to acoustic dipoles. Consequently, Gutin studied the following model. The air is subjected to forces distributed over the disk swept out by the propeller; each point on the disk experiences a constant thrust and torque when a blade passes that point, and no forces at other times. This system of forces is Fourier analysed, and the Fourier components of the sound field obtained. This model essentially yields an analysis of the sound produced by a force, or acoustic dipole, of constant absolute strength rotating in a circle. The sound is composed of a series of discrete tones, whose frequencies are multiples of the blade passage frequency. Both in quality and quantity the sound predicted by this theory is in agreement with the experimental evidence of Gutin's time.

The Gutin model has survived to the present time with very little change. Garrick and Watkins [2] have extended Gutin's analysis to account for the forward motion of the propeller, and Lowson [3], using more modern analytical techniques, has re-obtained Gutin's results, and also extended them to helicopter main rotor noise. On the other hand, the nature of the sound produced by modern fans and axial compressors, with their large number of blades and higher rotational speeds, bears very little resemblance to Gutin's model. The noise generated by modern rotating machinery is composed of broad-band noise distributed over a very wide spectrum, plus a series of superimposed discrete tones at multiples of the blade passage

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frequency, and sometimes of the lower disk rotation frequency. The intensity of the broad-band noise tends to follow a sixth power law of some typical velocity, and this clearly indicates that it is generated by random fluctuations in blade forces. These in turn are caused by such factors as the irregular shedding of vortices at the blade trailing edge (Lilley [4], Bragg and Bridge [5]) and the interaction of the blade with patches of turbulence in the oncoming stream. Gutin's analysis is clearly inapplicable to such situations, and to date no extension of his theory has been made that derives the general features of the noise from the random forces on a rotating blade. The discrete tones are usually taken to be generated by the mean force operating as in Gutin's model and by periodic inhomogeneities in the flow field, but the theoretically predicted level of these tones for single-stage fans usually falls short of experimental observation.

Both the work of Lilley [4] and Sharland [6] has gone a long way to close the gap between theory and experiment, and Hulse *et al.* [7] have made a major contribution in recognizing that imperfect propagation of the sound can account for substantial amplification of the blade passage frequency sound. None the less, there exists no rigorous theoretical treatment which is compatible with experimental evidence, and an attempt to close this gap is described below. In section 2 it is argued that acoustic quadrupoles not considered in the Gutin model may be important as sound sources, and their effect upon discrete tone generation is discussed in sections 3 and 4. Finally, section 5 is devoted to developing the formulae necessary to analyse the spectrum of a rotating multipole of arbitrary strength.

2. SOUND SOURCES IN MANY-BLADED ROTORS

The work of Gutin leads to an emphasis being placed on the distribution of pressure over the blade surface as being the prime source of propeller noise. The fluid is always treated as a perfect acoustical medium, propagating the pressure fluctuations generated by the blade as sound waves. The mathematical formulation of this outlook is to assume that the pressure satisfies the homogeneous wave equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \frac{\partial^2 p}{\partial x_i^2} = 0, \tag{1}$$

and is to be solved for in a region exterior to the blade, in terms of the given surface pressure distribution. As a mathematical boundary value problem, this is associated with the name of Kirchhoff, but as a description of propeller noise generation, it is natural to call this the Gutin formulation. Of course, Gutin did not set up the problem in this way, nor has it yet been tackled from this viewpoint, possibly because no correct solution of the Kirchhoff problem with moving boundaries has been published.

In reality, however, a fluid only behaves as a perfect acoustical medium if the fluid velocity is everywhere small compared with the speed of sound. Since surfaces moving at speeds comparable with the speed of sound will induce fluid velocities of the same magnitude, it is not valid to treat the fluid as a perfect acoustical medium. A more exact specification must be sought. Such a specification is supplied by Lighthill's work on aerodynamic noise [8, 9]. This shows that a correct equation describing the behaviour of the fluid is the forced wave equation for the density,

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{2}$$

where $T_{ij} = \rho u_i u_j + p_{ij} - c^2 \rho \delta_{ij}$. This equation means that the real fluid behaves like an ideal acoustical medium, which contains quadrupole sound sources of strength density T_{ij}. Thus Gutin's specification of the fluid is incomplete, but it preceded Lighthill's formulation by 20 years. These extra quadrupoles, which represent the sound generated by the fluid flow around the blade, are the only major difference between the two formulations. For although the two governing equations also differ in that one determines the pressure and the other the density, for an ideal fluid these are simply related by $\Delta p = c^2 \Delta \rho$ and so in Gutin's formulation the pressure could be replaced by the density.

From the mathematical point of view, the solution of Lighthill's equation consists of a particular integral plus a complementary function. The particular integral represents the sound generated by the distribution of quadrupole sources, and the complementary function represents the effects of the boundaries, and is identical with the solution obtained from Gutin's formulation. His work implies that the effect of the boundaries is equivalent to a surface distribution of acoustic dipoles whose strength equals the force P_i exerted on the fluid by the surface element. This is precisely the result obtained by Curle [10] from a formal solution of Lighthill's equation, and although his result is only valid for stationary surfaces, the authors' work [11] shows it to remain true when the surfaces are in motion; the boundaries are now equivalent to a moving distribution of dipoles. To summarize, theoretical considerations show that the sound produced by a propeller or fan can be regarded as being generated by quadrupoles of strength density T_{ij} distributed throughout the volume exterior to the blade, plus dipoles of strength density P_i distributed over the blade surface; theoretical work to date has ignored the quadrupole sources, and as will be shown, has consequently missed what seems to be the most efficient generator of discrete-tone fan noise in many-bladed machines.

It should be emphasized at this point that the statement that quadrupoles are less efficient than dipoles as sound sources cannot be invoked to dismiss the quadrupole sound; such an argument is only valid if a typical Mach number is small, but how small it must be in a given situation can only be determined after analysis. The quadrupole sound may depend upon other large factors which swamp the effect of Mach number. It cannot be assumed *a priori* that the sound produced by the quadrupoles is negligible compared with the dipole sound.

The importance of the quadrupoles as a source of broad-band noise is difficult to assess. This is because broad-band noise is generated by the random fluctuations of the sound sources, and no published analysis predicts the general characteristics of broad-band noise from a model of fluctuating sources in a rotational motion. Consequently our discussion of this point must be restricted to a few general remarks. A dimensional analysis of stationary sources shows that fluctuating forces generate a sound intensity that varies as the sixth power of a typical velocity, whereas fluctuations in the quadrupole strength produces an eighth power variation. The work in section 5 suggests that such dimensional variations are also valid for rotating sources, and so the experimentally observed sixth power dependence indicates that the fluctuating forces are the dominant source of broad-band noise. A possible explanation for this is that the flow induced around a blade by its motion is mainly laminar, the only significantly turbulent region being the blade wake. Thus, there is only a small distribution of quadrupoles with a randomly fluctuating strength, and the noise it generates is negligible compared with that produced by the force fluctuations. The presence of further rotors or stator rows increases the unsteadiness of the flow, but it is thought that such interactions between fixed and moving blade rows produce essentially periodic variations in the flow, and not a high level of genuine turbulence. Such periodic variations only contribute to the discrete tones. Thus, it appears likely that the quadrupoles are not a significant source of broad-band noise.

3. DISCRETE TONE GENERATION BY THE POTENTIAL FLOW FIELD

The situation for the discrete tones is different. If the fluid temperature remains nearly constant throughout the motion, the quadrupole strength density T_{ij} can be approximated by the Reynolds stress $\rho u_i u_j$. The flow field set up by the blade motion has particle speeds near the blade surface of the same order as the blade speed U, and consequently the quadrupole source strength in this region is of order ρU^2. The quadrupoles are of this magnitude in a region surrounding the blade and rotating with it. Just as the mean force rotating in a circle produces pure tones, so does the mean Reynolds stress. Because the mean surface pressure is also of magnitude ρU^2, the essential difference between the two terms is contained in their dipole-quadrupole character. To obtain an idea of their relative importance, a simplified situation will be studied.

In general, the sound field is generated by dipoles and quadrupoles distributed over the surface and throughout the volume, respectively. The central simplification in this model is to assume that these distributed sources can be replaced by a single rotating point source whose strength is equal to the integrated strength of the distributed sources. Such an assumption is difficult to justify but without it any analysis is very difficult, and requires detailed knowledge of the flow around the blade. It is further assumed that the two types of source rotate in the same circle. Only the axial component of the force (the thrust) and the longitudinal quadrupole aligned along the axis will be considered. Expressions for the sound fields of rotating point dipoles and quadrupoles of constant strength have been given by Lowson [3], and can be analysed into a Fourier series, whence the following expressions are obtained. For the far-field sound pressure of a dipole, the absolute magnitude of the nth harmonic (based on *rotational* frequency) is given by

$$D_n = \frac{n\Omega x T}{2\pi c r^2} J_n\left(\frac{nMy}{r}\right). \tag{3}$$

Here Ω is the rotational angular velocity, and T is the dipole strength equal to the total thrust exerted by a single blade. M is the rotational Mach number $\Omega R/c$, R being the radius of the circle. x and y are components of the distance r from the centre of rotation to the observer; x is the axial component, and y is the component in a direction perpendicular to the axis lying in the plane defined by the axis and the observer. For a quadrupole of strength Q, which equals the integrated mean Reynolds stress, the equivalent harmonic content is

$$Q_n = \frac{n^2 \Omega^2 x^2 Q}{2\pi c^2 r^3} J_n\left(\frac{nMy}{r}\right). \tag{4}$$

If the blade span is s and its chord l, then a representative area is sl, and volume sl^2. The total thrust T can be expressed as a thrust coefficient C_T multiplied by $\rho U^2 sl$, and similarly Q is expressed as $C_Q \rho U^2 sl^2$. The coefficients C_T and C_Q are both of order one. For a rotor with B equally spaced blades, the sound pattern of each blade is out of phase with those of other blades, and all harmonics which are not multiples of B cancel. For the remaining harmonics $n = mB$, and the ratio of quadrupole and dipole components of the sound field is

$$\frac{mB\Omega l}{c}\frac{x}{r}\frac{C_Q}{C_T} = mBM\frac{l}{R}\frac{x}{r}\frac{C_Q}{C_T}. \tag{5}$$

Although this ratio contains the Mach number, it also contains mB, and for a system with a large number of blades, this ratio may be large, especially for the higher harmonics. As an illustration, let us compute the magnitude of this ratio for the propeller quoted by Gutin, and for a modern engine compressor. The figures are only tentative, as chord size never seems to be recorded, and the effective radius R is not known. It is assumed that C_Q/C_T is of order unity. For Gutin's two-bladed propeller, values of 0·2 for l/R and 0·5 for M seem reasonable; for an observer at an angle of 30° from the axis listening to the fundamental note, this ratio is 0·17. Thus for Gutin's propeller, it appears that the quadrupoles are negligible as sources of discrete tones below about the tenth harmonic. On the other hand, for a compressor with 40 blades, figures of 0·2 for l/R and 0·8 for M are possible, and then the ratio is about 5·5 for the fundamental and increases in direct proportion to the harmonic number. This suggests that the mean Reynolds stress may be an important source of discrete tones for many-bladed compressors. However, it is to be emphasized that the relative importance of the two sources depends upon the observer's position, and factors such as C_Q/C_T whose magnitudes are largely unknown. Also, only a single component of each source has been analysed. Thus, the numbers derived above should only be taken to suggest that in many-bladed machinery both the surface forces and the Reynolds stresses can act through the same mechanism of sound generation to comparable effect.

But the foregoing analysis hopelessly underestimates the experimentally measured field near the axis of rotation. The predicted harmonic content depends on the Bessel function $J_{mB}(mBMy/r)$. These functions are extremely small when the order mB is large, especially near the axis of rotation; at 30° to the axis of a 40-bladed rotor moving at $M = 0{\cdot}8$ it is of magnitude 10^{-13} an impossibly low value. The Bessel functions are an unavoidable consequence of a source rotating in a circle; they are the remains of a large cancellation that is imperfect as a result of retarded time differences. To avoid these functions, a source must be sought that fluctuates at the blade passage frequency, but which does not rotate in a circle.

4. TURBULENCE AS A SOURCE OF DISCRETE TONES

The realization that the quadrupoles are important sound sources gives some scope for finding an alternative mechanism of discrete tone generation. The Reynolds stress quadrupoles are of strength density $\rho u_i u_j$ and distributed throughout the volume exterior to the blades. The flow through a rotor disk is principally a potential flow, but with a certain level of turbulence superimposed upon it. This turbulence may originate from upstream obstacles in the flow, or from the rotor blades themselves. To bring out this feature, the velocity u_i is split into a potential flow velocity U_i and a turbulence velocity v_i; the quadrupole strength then becomes

$$\rho u_i u_j = \rho U_i U_j + \rho(U_i v_j + U_j v_i) + \rho v_i v_j. \tag{6}$$

The first of these terms is the source effect of the potential field, and has already been discussed in the previous section. Similarly, the third term represents a purely turbulent source, and has also been mentioned, although its effect not properly assessed. However, the second term represents an interaction between the potential and turbulent fields, and as such has not yet been mentioned as a possible source of sound. In fact, the potential velocity may be further divided into a convection velocity through the rotor, and a purely oscillatory velocity induced by the repeated passage of rotor blades. It is with the interaction between the oscillatory and turbulence velocity fields that this section is concerned.

The properties of an interaction source term such as $\rho U_i v_j$ are a combination of those of its two components, and are deduced as follows. The potential velocity U_i can be assumed to fluctuate essentially sinusoidally at the blade passage (radian) frequency ΩB. It is of the order of the blade speed U for a distance of the order of the blade chord l either side of the rotor disk. Beyond this region, the potential velocity, and hence the interaction source, is negligible. On the other hand, the turbulence velocity v_i is random in space and time, and must be treated stochastically. The turbulence can be regarded as being composed of a number of eddies; the velocity being well correlated throughout each eddy. In general, these well correlated regions convect downstream with a velocity U_c. This convection velocity is not necessarily the downstream fluid velocity, but is the velocity of a frame of reference in which the turbulence changes most slowly. The time taken for an eddy of dimension L to pass a fixed point is clearly LU_c^{-1}, and in this time, the number of blades N that rotate past that point is

$$N=\frac{L}{U_c}\frac{\Omega B}{2\pi}=\frac{B}{2\pi}\frac{L}{R}\frac{M}{M_c}. \tag{7}$$

N represents the number of blades that chop through the eddy as it convects past the rotor. The magnitude of N depends upon various parameters, but it is instructive to proceed on the likely assumption that it is much greater than unity. The lifetime of an eddy is usually greater than the time it takes to convect past a given point LU_c^{-1}; the eddy travels more than its own diameter before losing its identity. Thus at a fixed point, the turbulence velocity changes in a time scale set by the convection time LU_c^{-1}, rather than the eddy lifetime, and so a typical (radian) frequency of this velocity is $2\pi U_c L^{-1}$. The condition that N is large then implies that the blade passage frequency is much greater than a typical turbulence frequency, and it follows that a product of the potential and turbulence velocities fluctuates at the higher blade passage frequency. Thus we see that the distributed interaction quadrupoles $\rho U_i v_j$ are of a significant magnitude in a region either side of the rotor, they can be grouped into coherent eddies, and fluctuate mainly at the blade passage frequency.

For a quadrupole source, the frequency spectrum observed in the far field is the spectrum of the quadrupole source strength multiplied by ω^4. In our situation, the quadrupole strength is $\rho U_i v_j$, and, as we have seen, this fluctuates mainly at the blade passage frequency. In fact it can be shown that the spectrum of such a product, one term of which varies sinusoidally at frequency ΩB, is just the spectrum of the other component shifted by ΩB. Thus the width of the source spectrum is the width of the spectrum of its turbulence component, but its mean frequency is shifted to ΩB. As the width of the turbulence spectrum is of the order of its typical frequency $2\pi U_c L^{-1}$, the bandwidth $\omega^{-1}\Delta\omega$ of the source spectrum is

$$\frac{\Delta\omega}{\omega}=\frac{2\pi U_c}{L\Omega B}=\frac{l}{N}. \tag{8}$$

Thus for large N, the source spectrum is very narrow, and centred on the blade passage frequency. The spectrum observed in the far field is this spectrum modified by ω^4, which can be regarded as constant throughout the narrow band. Thus the observed spectrum is also very narrow, and this mechanism is essentially a source of discrete tones. On the other hand if N is not large, the source spectrum is broader, and is further distorted by the factor ω^4; the sound is then relatively broad band, the maximum bandwidth being of order unity. Thus the nature of the radiated sound depends upon the magnitude of N; for large N the sound is blade passage tones, but for small N it is broad-band noise. Clearly, N increases with the number of blades, but it may require that M_c is small compared with M for it to be very large.

To assess the intensity of the sound generated by this mechanism, we must estimate the magnitude of the basic integral describing the sound field of a quadrupole distribution. One possible simplification is to neglect the retarded time differences across the eddy, but this is a bad assumption for the following reason. Although the time taken for the sound to travel across the eddy is a small fraction of the eddy lifetime, it is a large fraction of the periodic time of the potential velocity. Thus although points on opposite sides of an eddy have velocities that are well correlated instantaneously, the sound they produce is significantly out of phase when it reaches the observer. That is, the eddies are no longer acoustically compact. In this situation, the effective volumetric scale is established by the eddy area and the distance travelled by a sound wave in a characteristic period [12], and the best expression for the far-field sound pressure of a non-compact quadrupole distribution is

$$p(t)=\frac{1}{4\pi}\int\left[\frac{\partial^2(\rho U_i v_j)}{\partial y_i\,\partial y_i}\right]\frac{\mathrm{d}S c\,\mathrm{d}\tau}{r}. \tag{9}$$

Here the square brackets indicate that the integral is to be evaluated over the surface of the sphere S, $r = c(t-\tau)$.

The magnitude of the sound pressure produced by one eddy can be estimated from this integral. It is assumed that the eddy dimension L is the same as the blade chord length l. In the region either side of the rotor, the potential velocity is of order U and the turbulent velocity αU. A space derivative is equivalent to division by the chord length l. The eddy area is l^2, and the relevant characteristic time is $2\pi(\Omega B)^{-1}$, so that the sound pressure produced by a single eddy is of magnitude

$$p = \frac{\rho U^2}{l^2}\frac{\alpha}{4\pi r}\cdot\frac{l^2 c 2\pi}{\Omega B} = \frac{\rho U \alpha c R}{2rB}. \tag{10}$$

The eddy is a coherent source and so the sound power it generates is proportional to the square of this expression. The total volume of the region under discussion is $2\pi Rsl$, which thus contains $2\pi Rsl^{-2}$ turbulent eddies. The total mean square sound pressure generated by all the eddies is

$$\overline{p^2} = \frac{\rho^2 U^2 \alpha^2 c^2 R^2}{4r^2 B^2}\cdot\frac{2\pi Rs}{l^2} = \frac{\pi\rho^2 U^2 \alpha^2 c^2 R^3 s}{2B^2 r^2 l^2}. \tag{11}$$

An acoustical efficiency for this mechanism can be calculated to be

$$\eta = \frac{2\pi^2 \alpha^2}{C_D M B^3}\left(\frac{R}{l}\right)^3. \tag{12}$$

This equation is restricted to frequencies such that the wavelength of sound at the blade passage frequency is not significantly larger than the blade chord, so that it would be misleading to apply this result when both B and M are small. However, for high-speed many-bladed fans this formula is relevant. If the previously quoted values are used, and a 1% turbulence level is assumed, this efficiency turns out to be of the order of 10^{-5}. This figure appears very reasonable, but rather on the high side, but again the analysis has been very crude, and could easily be in error by a factor of ten.

One conclusion from the above analysis is inevitable. In situations where N is large, this mechanism is far more likely to be the cause of discrete tones near the axis of multibladed machinery than any Gutin type of mechanism. Equation (11) also shows that the sound intensity generated by this mechanism, whether it be discrete tone or broad band, varies as the square of the rotational Mach number, instead of the usual sixth power variation valid at low rotational speeds. It is evident from the above discussion that this mechanism operates when the blades chop through the wakes of upstream obstacles. Such wake-chopping situations are known to be powerful sources of discrete tones, but it is seen that it is not necessary to seek unsteadiness in the blade forces as the sole cause of high-amplitude discrete tones.

5. RADIATION FROM ROTATING MULTIPOLES AT FREQUENCIES NOT NECESSARILY RELATED TO ROTATION FREQUENCIES

The generation and propagation of sound is governed by the inhomogeneous wave equation

$$\frac{\partial^2 \rho}{\partial t^2} - c^2\frac{\partial^2 \rho}{\partial x_i^2} = Q(\mathbf{x},t). \tag{13}$$

Here $Q(\mathbf{x},t)$ is the source strength density at $(\mathbf{x},t)$ which may or may not be concentrated in the form of a delta function singularity. In the general aerodynamic noise problem with boundaries, the sources may include simple, dipole and quadrupole elements, so that

$$Q(\mathbf{x},t) = S(\mathbf{x},t) + \frac{\partial D_i(\mathbf{x},t)}{\partial x_i} + \frac{\partial^2 T_{ij}(\mathbf{x},t)}{\partial x_i \partial x_j}. \tag{14}$$

S, D_i and T_{ij} are respectively the strength density of the three source types. The solution of (13) in the distant radiation field can be written in either real time or spectral form:

$$\rho(\mathbf{x},t)=\frac{1}{4\pi c^2 r}\int\limits_v Q\left(\mathbf{y},t-\frac{r}{c}\right)\mathrm{d}\mathbf{y}, \tag{15}$$

$$\rho(\mathbf{x},\omega)=\frac{\mathrm{e}^{-2\pi i\omega r/c}}{4\pi c^2 r}\,Q\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right). \tag{16}$$

r is the distance separating source and observer and $\hat{\mathbf{r}}$ is the unit vector in the radiation direction. $\rho(\mathbf{x},\omega)$ is the generalized Fourier transform of $\rho(\mathbf{x},t)$:

$$\rho(\mathbf{x},\omega)=\int\limits_{-\infty}^{\infty}\rho(\mathbf{x},t)\,\mathrm{e}^{-2\pi i\omega t}\,\mathrm{d}t, \tag{17}$$

and $Q(\mathbf{k},\omega)$ is the wave number, frequency, spectral component of the source field, being the four-dimensional Fourier transform of $Q(\mathbf{x},t)$:

$$Q(\mathbf{k},\omega)=\iint Q(\mathbf{x},t)\,\mathrm{e}^{-2\pi i\mathbf{k}.\mathbf{x}}\,\mathrm{e}^{-2\pi i\omega t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{18}$$

In terms of the spectral components of the multipole strength,

$$Q(\mathbf{k},\omega)=S(\mathbf{k},\omega)+2\pi ik_i\,D_i(\mathbf{k},\omega)-(2\pi)^2\,k_i\,k_j\,T_{ij}(\mathbf{k},\omega) \tag{19}$$

and the particular component generating the acoustic wave at frequency ω propagating in the direction $\hat{\mathbf{r}}$ is

$$Q\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right)=S\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right)-2\pi i\frac{\omega}{c}\hat{\mathbf{r}}_i\,D_i\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right)-(2\pi)^2\frac{\omega^2}{c^2}\hat{r}_i\hat{r}_j T_{ij}\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right). \tag{20}$$

$\hat{r}_i$ is the component of the unit vector $\hat{\mathbf{r}}$ in the direction i.

The object of this section is to develop the theory of fan noise at frequencies which are not necessarily multiples of the blade passage frequency, and for this we can, without any loss of generality, regard the sources as concentrated at a point rotating in the $x_3=0$ plane at radius R with frequency Ω. Then $T_{ij}(\mathbf{x},t)$, $D_i(\mathbf{x},t)$ and $S(\mathbf{x},t)$ can be written symbolically as

$$\left.\begin{matrix}T_{ij}\\ D_i\\ S\end{matrix}\right\}(\mathbf{x},t)=q(\phi,t)\,\delta(\mathbf{x}-\mathbf{R}). \tag{21}$$

$q(\phi,t)$ is the time-dependent strength of the source which occupied angular position ϕ from the x-axis at time $t=0$, and $\mathbf{R}$ is the position vector of that source at time t. From equation (18), the four-dimensional Fourier transform of this source density is

$$\left.\begin{matrix}T_{ij}\\ D_i\\ S\end{matrix}\right\}(\mathbf{k},\omega)=\int\limits_{-\infty}^{\infty}q(\phi,t)\,\mathrm{e}^{-2\pi i.\mathbf{k},\mathbf{R}}\,\mathrm{e}^{-2\pi i\omega t}\,\mathrm{d}t. \tag{22}$$

Now $q(\phi,t)$ may be expressed in terms of its generalized Fourier transform (or spectral components) as,

$$q(\phi,t)=\int\limits_{-\infty}^{\infty}q(\phi,\alpha)\,\mathrm{e}^{2\pi i\alpha t}\,\mathrm{d}\alpha, \tag{23}$$

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and the radiating component of the source spectrum becomes

$$\left.\begin{matrix} T_{ij} \\ D_i \\ S \end{matrix}\right\}\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right) = \iint\limits_{-\infty}^{\infty} q(\phi,\alpha)\,\mathrm{e}^{-2\pi i t(\omega-\alpha)}\,\mathrm{e}^{2\pi i(\omega/c)R\sin\theta\cos(2\pi\Omega t+\phi)}\,\mathrm{d}t\,\mathrm{d}\alpha. \tag{24}$$

Here $\mathbf{k}.\mathbf{R}$ has been set equal to $-(\omega/c)\hat{\mathbf{r}}.\mathbf{R}$ which is $-(\omega/c)R\sin\theta\cos(2\pi\Omega t+\phi)$, θ being the angle at which the wave is propagating measured relative to the fan axis, which is $\theta = 0$.

The time (t) integration can now be performed (Jones [13], p. 137):

$$\left.\begin{matrix} T_{ij} \\ D_i \\ S \end{matrix}\right\}\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right) = \int\limits_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q(\phi,\alpha)\,\mathrm{e}^{-in(\pi/2)}\,\mathrm{e}^{+i\phi[(\omega-\alpha)/\Omega]}\,J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right)\delta\left(\frac{\omega-\alpha}{\Omega}-n\right)\frac{\mathrm{d}\alpha}{\Omega}$$

$$= \sum_{n=-\infty}^{\infty} q(\phi,\omega-n\Omega)\,\mathrm{e}^{-in(\pi/2-\phi)}\,J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right), \tag{25}$$

where J_n is the Bessel function of the first kind.

The Gutin theory is simply derived as a special case of this result. That deals with the field established by point sources of constant absolute strength. If we take by way of example the field induced in the plane $x_2 = 0$ by the torque dipole, then only D_1 need by retained. If D_1 is expressed as

$$D_1(\mathbf{x},t) = D\cos\left(2\pi\Omega t+\phi-\frac{\pi}{2}\right)\delta(\mathbf{x}-\mathbf{R}), \tag{26}$$

then for this source

$$q(\phi,\alpha) = \frac{D}{2}\{\mathrm{e}^{i(\phi-\pi/2)}\,\delta(\alpha-\Omega)+\mathrm{e}^{-i(\phi-\pi/2)}\,\delta(\alpha+\Omega)\}, \tag{27}$$

and

$$D_1\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right) = \frac{D}{2}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-i(n+1)(\pi/2-\phi)}\,\delta(\omega-(n+1)\Omega)\,J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right)+$$

$$+\frac{D}{2}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-i(n-1)(\pi/2-\phi)}\,\delta(\omega-(n-1)\Omega)\,J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right), \tag{28}$$

$$= -\frac{Dc}{2\pi\omega R\sin\theta}\sum_{n=-\infty}^{\infty} n\mathrm{e}^{-in(\pi/2-\phi)}\,\delta(\omega-n\Omega)\,J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right). \tag{29}$$

Equation (29) is obtained by re-ordering the summations in (28), and employing a well-known Bessel function recurrence relation. This source field generates a radiation field given by equations (16) and (20), equal to

$$\frac{\mathrm{e}^{-2\pi i\omega r/c}}{4\pi c^2 r}\left(-2\pi i\frac{\omega}{c}\sin\theta\right)D_1\left(-\frac{\omega}{c}\hat{\mathbf{r}},\omega\right), \tag{30}$$

so that the result for the Gutin theory becomes

$$\rho(\mathbf{x},\omega) = \frac{\mathrm{e}^{-2\pi i\omega r/c}}{4\pi c^2 r}\frac{iD}{R}\sum_{n=-\infty}^{\infty} n\,\mathrm{e}^{-in(\pi/2-\phi)}\,\delta(\omega-n\Omega)\,J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right). \tag{31}$$

This is the well-known result for the spectrum function, showing the spectrum of each dipole to be discrete at harmonics of the disk rotation frequency. If there are B identical sources spaced in an exactly regular array around a circle at angular intervals $2\pi/B$, then the total effect of all sources will involve the sum

$$\left.\begin{aligned}\sum_{\zeta=1}^{B} e^{i\zeta n2\pi/B} &= B; \quad n = mB\\ &= 0; \quad n \neq mB,\end{aligned}\right\} \tag{32}$$

where m is an integer. The magnitude of the spectral level generated by B sources then becomes

$$|\rho(\mathbf{x},\omega)| = \left|\frac{D}{4\pi c^2 r}\frac{B^2}{R}\sum_{m=-\infty}^{\infty} m\delta(\omega - mB\Omega) J_{mB}\left(\frac{2\pi\omega R\sin\theta}{c}\right)\right|, \tag{33}$$

which is Gutin's discrete spectrum at harmonics of the blade passage frequency. Of course imperfect duplication of the source or blade spacing would nullify the above summation, and leave a finite level at harmonics of the disk rotation frequency.

We return now to the general case and give three formulae for the spectral components of the distant radiation field in terms of the spectral level of the source strength. The first is for a point monopole with spectrum $s(\phi,\alpha)$, the second for a point dipole with spectrum $d_i(\phi,\alpha)$ and the third for a point quadrupole with spectrum $t_{ij}(\phi,\alpha)$. By spectrum we mean, of course, the generalized Fourier transform of the time-dependent source strength. In each case the point source is rotating in a circular path of radius R at frequency Ω and occupies angular position ϕ at time $t = 0$.

(i) Monopole:

$$\rho(\mathbf{x},\omega) = \frac{e^{-2\pi i\omega r/c}}{4\pi c^2 r}\sum_{n=-\infty}^{\infty} s(\phi,\omega - n\Omega)\, e^{-in(\pi/2-\phi)} J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right). \tag{34}$$

(ii) Dipole:

$$\rho(\mathbf{x},\omega) = -\frac{e^{-2\pi i\omega r/c}}{4\pi c^2 r}\frac{2\pi i\omega}{c}\hat{r}_i\sum_{n=-\infty}^{\infty} d_i(\phi,\omega - n\Omega)\, e^{-in(\pi/2-\phi)} J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right). \tag{35}$$

(iii) Quadrupole:

$$\rho(\mathbf{x},\omega) = -\frac{e^{-2\pi i\omega r/c}}{4\pi c^2 r}(2\pi)^2\frac{\omega^2}{c^2}\hat{r}_i\hat{r}_j\sum_{n=-\infty}^{\infty} t_{ij}(\phi,\omega - n\Omega)\, e^{-in(\pi/2-\phi)} J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right). \tag{36}$$

All these forms involve the spectral level which is a generalized function that cannot be given a unique meaning. To make these expressions practically useful, one must form power spectral density relations to predict statistics of the radiation field in terms of statistics of the source strength. This is done by multiplying both sides of these equations by their conjugates, averaging, and normalizing by a factor common to both sides whose magnitude need not be considered and can be taken as unity. We will deal particularly with the dipole equation to develop a formula relating the power spectral density of a time-dependent dipole strength, such as might be induced by vortex shedding on a fan blade, to the power spectral density of the density fluctuation in the distant radiation field, $P(\mathbf{x},\omega)$.

$$P(\mathbf{x},\omega) = \frac{\omega^2}{4c^6 r^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\overline{d_r(\phi,\omega - n\Omega)\, d_r(\phi,-\omega + m\Omega)}\, e^{-i(n-m)(\pi/2-\phi)} \times$$
$$\times J_n\left(-\frac{2\pi\omega R\sin\theta}{c}\right) J_m\left(-\frac{2\pi\omega R\sin\theta}{c}\right). \tag{37}$$

Here d_r has been written for the dipole component in the particular radiation direction

$\hat{r}_i d_i$. If we now assume that all Fourier elements are uncorrelated unless they are conjugates, and this is so in a stationary field, we can reduce the double sum to a single summation over n:

$$\left.\begin{aligned} \overline{d_r(\phi\,\alpha)\,d_r(\phi,\beta)} &= D_r(\phi,\alpha); \quad \alpha = -\beta \\ &= 0; \qquad\qquad\quad \alpha \neq -\beta \end{aligned}\right\} \tag{38}$$

where $D_r(\phi, \alpha)$ is the power spectral density of the dipole strength in the radiation direction. Then

$$P(\mathbf{x}, \omega) = \frac{\omega^2}{4c^6 r^2} \sum_{n=-\infty}^{\infty} D_r(\phi, \omega - n\Omega) J_n^2\left(-\frac{2\pi\omega R \sin\theta}{c}\right). \tag{39}$$

This formula can be simplified in two special cases when the dipole spectral density is not a function of n and the summation can be carried out exactly. First, if the source is not rotating, $\Omega = 0$ and the Bessel function sum is unity,

$$P(\mathbf{x}, \omega)|_{\Omega=0} = \frac{\omega^2}{4c^6 r^2} D_r(\phi, \omega). \tag{40}$$

This is the well-known radiation field of a stationary point dipole. Second, if the dipole spectrum is flat, its variation is that of white noise. Again the Bessel functions sum is unity to give a radiation field power spectral density proportional to ω^2:

$$P(\mathbf{x}, \omega) = \frac{\omega^2}{4c^6 r^2} D_r(\phi). \tag{41}$$

This result is evidently unaffected by blade rotation.

In general no great simplification of the above result (39) is possible. It can be seen that a source at frequency α beats with harmonics of the disk rotation frequency $n\Omega$ to radiate at frequency $(\alpha + n\Omega)$. A dipole with a discrete spectrum radiates a discrete spectrum at these beat frequencies. A dipole with a narrow-band spectrum radiates in narrow bands about these beat frequencies, so that no definite tone is heard. On the other hand, a spectrum that is flat over a wide frequency range tends to the white noise case discussed above, where there is no hint of a tone. The most general conclusion therefore seems to be that the width of the spectral peaks which are radiated at beat frequencies are directly proportional to the bandwidth of the dipole frequency spectrum, and the strength of the narrow-band content is inversely proportional to the spectral bandwidth.

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