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# Vibration analysis of laminated composite rectangular plates with general boundary conditions

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**ABSTRACT**

**In this investigation, the free vibration analysis of laminated composite rectangular plates with general boundary conditions is performed with a modified Fourier series method. Vibration characteristics of the plates have been obtained via an energy function represented in the general coordinates, in which the displacement and rotation in each direction is described as an improved form of double Fourier cosine series and several closed-form auxiliary functions to eliminate any possible jumps and boundary discontinuities. All the expansion coefficients are then treated as the generalized coordinates and determined by Rayleigh-Ritz method. The convergence and reliability of the current method are verified by comparing with the results in the literature and those of Finite Element Analysis. The effects of boundary conditions and geometric parameters on the frequencies are discussed as well. Finally, numerous new results for laminated composite rectangular plates with different geometric parameters are presented for various boundary conditions, which may serve as benchmark solutions for future research.**

## 1 INTRODUCTION

Laminated composite rectangular plates are novel structural components widely used in various engineering fields, such as acoustic, aircraft and automobile manufacturing[1, 2]. And in some case they are unavoidably suffered from dynamic loads, which may lead to fatigue damage and severe reductions of the structures[3, 4]. Therefore, it is essential to understand the free vibration characteristics of laminated plate structures[5].

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To deal with the vibration problem of laminated plates, a lot of efficient methods have been devoted to the study for many years, such as TSDT[6], Ritz method[7], wave propagation approach[8], meshfree method[9], etc.

To improve the accuracy and convergence of the aforementioned methods, a modified Fourier series solution is proposed for the free vibration of laminated composite plates with general boundary conditions and linear Winkler and Pasternak foundation. Displacement and rotation in each direction is described as an improved form of double Fourier cosine series and several closed-form auxiliary functions, and exact solutions are obtained by Rayleigh-Ritz method. The reliability of the proposed method is demonstrated by comparing with the results in the literature and those of Finite Element Analysis. On this basis, numerous new results for laminated composite plates with different geometric parameters and angle-plies are presented. In addition, the effects of boundary conditions, geometric parameters and Winkler and Pasternak foundation parameters on the frequencies are also illustrated.

## 2 THEORETICAL FORMULATION

### 2.1 Model Description

A laminated composite rectangular plate with length $a$, width $b$, and total thickness of $h$ is depicted in Fig 1. To describe the plate clearly, we introduce the following Cartesian coordinate system: the displacements of the plate in the $x$, $y$ and $z$ directions are denoted by $u$, $v$ and $w$, respectively. The laminated rectangular plate is assumed to be $N$ composite layers. The frequently encountered boundary conditions are realized by three independent springs, which is translational, rotational and torsional springs. By assigning the stiffness of the springs at proper values can be equal to imposing different boundary forces on the mid-surface of the plate. $K_\varphi^x$, $K_\varphi^y$, $k_\varphi^u$, $k_\varphi^v$ and $k_\varphi^w$ are used to indicate the stiffness of the springs at $x$=0, $a$ and $y$=0, $b$, respectively[10]. For the elastic foundation, $K_w$ and $K_s$ are defined as linear Winkler and Pasternak foundation parameters.

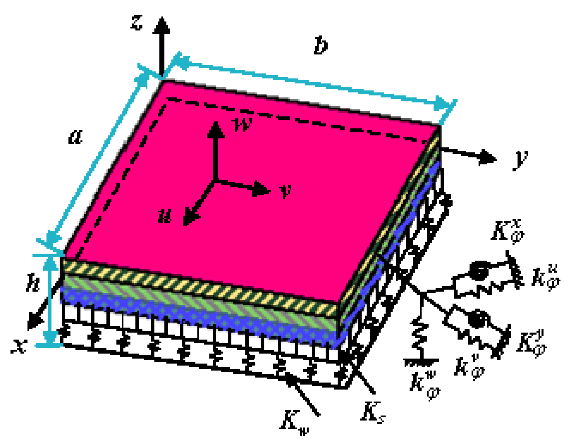


*Fig. 1 – Boundary restraints, Winkler and Pasternak foundation of laminated composite rectangular plate.*

### 2.2 Stress-Strain Relations and Stress Resultants

Based on the plate model presented above and the first-order shear deformation theory[11, 12], the displacement fields of laminated composite plates are:

$$
\begin{aligned}
U(x,y,z,t) &= u(x,y,t) + z\phi_x(x,y,t) \\
V(x,y,z,t) &= v(x,y,t) + z\phi_y(x,y,t) \\
W(x,y,z,t) &= w(x,y,t)
\end{aligned} \tag{1}
$$

where $u$, $v$ and $w$ are the middle surface displacements of the plate and $t$ is the time variable. $\phi_x$ and $\phi_y$ represent the rotations of transverse respect to $y$-axes and $x$-axes, respectively. For the laminated composite plates, the strains-displacement can be defined as:

$$
\left\{\varepsilon_x \varepsilon_y \gamma_{xy}\right\}^{\mathrm{T}} = \left\{\varepsilon_x^0 + z\chi_x \; \varepsilon_y^0 + z\chi_y \; \gamma_{xy}^0 + z\chi_{xy}\right\}^{\mathrm{T}} = \left\{\frac{\partial u}{\partial x} \; \frac{\partial v}{\partial y} \; \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}^{\mathrm{T}} + z\left\{\frac{\partial \phi_x}{\partial x} \; \frac{\partial \phi_y}{\partial y} \; \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right\}^{\mathrm{T}} \tag{2}
$$

$$
\left\{\gamma_{xz} \; \gamma_{yz}\right\}^{\mathrm{T}} = \left\{\gamma_{xz}^0 \; \gamma_{yz}^0\right\}^{\mathrm{T}} = \left\{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \; \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right\}^{\mathrm{T}} = \left\{\phi_x + \frac{\partial w}{\partial x} \;\; \phi_y + \frac{\partial w}{\partial y}\right\}^{\mathrm{T}} \tag{3}
$$

where the normal and shear strains in the $x$, $y$ and $z$ directions are denoted by $\varepsilon_x$ $\varepsilon_y$ and $\gamma_{xy}$, respectively. $\gamma_{xz}$ and $\gamma_{yz}$ are the transverse shear strains, which are constants through the thickness. And the matrix can be simplified to the following forms:

$$
\boldsymbol{\varepsilon} = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^{\mathrm{T}} \quad \boldsymbol{\chi} = \left(\frac{\partial \phi_x}{\partial x}, \frac{\partial \phi_y}{\partial y}, \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^{\mathrm{T}} \quad \boldsymbol{\gamma} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^{\mathrm{T}} \tag{4}
$$

According to Hooke's law[13], the stress-strain relations of the plates are written as:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q_{11}^k}(z) & \overline{Q_{12}^k}(z) & 0 & 0 & \overline{Q_{16}^k}(z) \\ \overline{Q_{12}^k}(z) & \overline{Q_{22}^k}(z) & 0 & 0 & \overline{Q_{26}^k}(z) \\ 0 & 0 & \overline{Q_{44}^k}(z) & \overline{Q_{45}^k}(z) & 0 \\ 0 & 0 & \overline{Q_{45}^k}(z) & \overline{Q_{55}^k}(z) & 0 \\ 0 & 0 & 0 & 0 & \overline{Q_{66}^k}(z) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_k \tag{5}
$$

By carrying the integration of stresses over the plate thickness, the force and moment resultants are obtained as

$$
(N_x, N_y, N_{xy})^{\mathrm{T}} = \int_{-h/2}^{h/2} \left[\sigma_x, \sigma_y, \tau_{xy}\right] dz \tag{6}
$$

$$
(M_x, M_y, M_{xy})^{\mathrm{T}} = \int_{-h/2}^{h/2} \left[\sigma_x, \sigma_y, \tau_{xy}\right] z dz \tag{7}
$$

$$
(Q_x, Q_y)^{\mathrm{T}} = \int_{-h/2}^{h/2} \left[\tau_{xz}, \tau_{yz}\right] dz \tag{8}
$$

where $N_x$, $N_y$ and $N_{xy}$ denote the normal and shear force resultants, respectively. $M_x$, $M_y$ and $M_{xy}$ are the bending and twisting moment resultants. $Q_x$ and $Q_y$ are the transverse shear force resultants. Performing the integration operation in Equations (6)-(8) yields:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \boldsymbol{\varepsilon} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \boldsymbol{\chi} \tag{9}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \boldsymbol{\varepsilon} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \boldsymbol{\chi} \tag{10}$$

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \kappa \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \boldsymbol{\gamma} \tag{11}$$

Here, the shear correction factor is denoted as $\kappa$. The extensional stiffness coefficients $A_{ij}$, extensional-bending stiffness coefficients $B_{ij}$ and bending stiffness coefficients $D_{ij}$ can be expressed as:

$$\left\{A_{ij}, B_{ij}, D_{ij}\right\} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} \left\{1, z, z^2\right\} dz \tag{12}$$

## 2.3 Energy Functions

In this section, the modified Fourier series method of Rayleigh-Ritz method is presented. In the Rayleigh-Ritz method, a displacement associated with undetermined coefficients is assumed and substituted into the Lagrangian energy function[14]. Then by minimizing the total expression and making them equal to zero, the coefficients in the displacement are determined. Finally, a series of equations can be summed up in matrix form as a standard characteristic equation. And the desired frequencies and modes can be obtained easily by solving the standard characteristic equation.

For free vibration analysis, the Lagrangian energy function of the laminated composite plates can be written as:

$$L = T - U_s - U_{sp} - U_f + W_e \tag{13}$$

The strain energy $U_s$ and kinetic energy T of the plates during vibration are given by:

$$U_s = \frac{1}{2} \int_0^a \int_0^b \left\{ N_x \varepsilon_x^0 + N_y \varepsilon_y^0 + N_{xy} \gamma_{xy}^0 + M_x \chi_x + M_y \chi_y + M_{xy} \chi_{xy} + Q_x \gamma_{xz}^0 + Q_y \gamma_{yz}^0 \right\} dydx \tag{14}$$

$$T = \frac{1}{2} \int_0^a \int_0^b \left\{ I_0 \left(\frac{\partial u}{\partial t}\right)^2 + I_0 \left(\frac{\partial v}{\partial t}\right)^2 + I_0 \left(\frac{\partial w}{\partial t}\right)^2 + I_2 \left(\frac{\partial \phi_x}{\partial t}\right)^2 + 2I_1 \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial \phi_x}{\partial t}\right) + 2I_1 \left(\frac{\partial v}{\partial t}\right)\left(\frac{\partial \phi_y}{\partial t}\right) + I_2 \left(\frac{\partial \phi_y}{\partial t}\right)^2 \right\} dydx \tag{15}$$

The work done by the external forces ($q_x$, $q_y$, $q_z$) and external couples ($m_x$, $m_y$) in the middle surface is:

$$W_e = \int_0^a \int_0^b \left\{ q_x u + q_y v + q_z w + m_x \phi_x + m_y \phi_y \right\} dydx \tag{16}$$

And the deformation strain energy in the boundary springs can be defined as:

$$U_{sp} = \frac{1}{2}\int_0^a \left\{ \begin{array}{l} [k_{y_0}^u u^2 + k_{y_0}^v v^2 + k_{y_0}^w w^2 + K_{y_0}^x \phi_x{}^2 + K_{y_0}^y \phi_y{}^2]_{y=0} \\ +[k_{y_1}^u u^2 + k_{y_1}^v v^2 + k_{y_1}^w w^2 + K_{y_0}^x \phi_x{}^2 + K_{y_0}^y \phi_y{}^2]_{y=b} \end{array} \right\} dx + \frac{1}{2}\int_0^a \left\{ \begin{array}{l} [k_{x_0}^u u^2 + k_{x_0}^v v^2 + k_{x_0}^w w^2 + K_{x_0}^x \phi_x{}^2 + K_{x_0}^y \phi_y{}^2]_{x=0} \\ +[k_{x_1}^u u^2 + k_{x_1}^v v^2 + k_{x_1}^w w^2 + K_{x_1}^x \phi_x{}^2 + K_{x_1}^y \phi_y{}^2]_{x=a} \end{array} \right\} dy \tag{17}$$

Subsequently, the strain energy based on the Winkler and Pasternak foundations is:

$$U_f = \frac{1}{2}\int_0^a \int_0^b \left\{ k_w w^2 + k_s \left( \frac{\partial w}{\partial x} \right)^2 + k_s \left( \frac{\partial w}{\partial y} \right)^2 \right\} dydx \tag{18}$$

## 2.4 Governing Equations and Admissible Displacement Functions

The governing equations for the plates can be obtained by applying Hamilton's principle. Regardless of boundary conditions, each displacement and rotation field of the laminated plates can be described by the two-dimensional modified Fourier technique:

$$u(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn} \cos\lambda_m x \cos\lambda_n y + \sum_{l=1}^{2}\sum_{n=0}^{N} a_l^n \zeta_l^a(x) \cos\lambda_n y + \sum_{l=1}^{2}\sum_{m=0}^{M} b_l^m \zeta_l^b(y) \cos\lambda_m x \tag{19}$$

$$v(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} B_{mn} \cos\lambda_m x \cos\lambda_n y + \sum_{l=1}^{2}\sum_{n=0}^{N} c_l^n \zeta_l^a(x) \cos\lambda_n y + \sum_{l=1}^{2}\sum_{m=0}^{M} d_l^m \zeta_l^b(y) \cos\lambda_m x \tag{20}$$

$$w(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} C_{mn} \cos\lambda_m x \cos\lambda_n y + \sum_{l=1}^{2}\sum_{n=0}^{N} e_l^n \zeta_l^a(x) \cos\lambda_n y + \sum_{l=1}^{2}\sum_{m=0}^{M} f_l^m \zeta_l^b(y) \cos\lambda_m x \tag{21}$$

$$\phi_x(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} D_{mn} \cos\lambda_m x \cos\lambda_n y + \sum_{l=1}^{2}\sum_{n=0}^{N} g_l^n \zeta_l^a(x) \cos\lambda_n y + \sum_{l=1}^{2}\sum_{m=0}^{M} h_l^m \zeta_l^b(y) \cos\lambda_m x \tag{22}$$

$$\phi_y(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{N} E_{mn} \cos\lambda_m x \cos\lambda_n y + \sum_{l=1}^{2}\sum_{n=0}^{N} i_l^n \zeta_l^a(x) \cos\lambda_n y + \sum_{l=1}^{2}\sum_{m=0}^{M} j_l^m \zeta_l^b(y) \cos\lambda_m x \tag{23}$$

where $\lambda_m = m\pi / a$ and $\lambda_n = n\pi / b$. $M$ and $N$ are the truncation numbers of variables $x$ and $y$. $A_{mn}$, $B_{mn}$, $C_{mn}$, $D_{mn}$ and $E_{mn}$ are the Fourier expansion coefficients of the cosine Fourier series. $a_l^n$, $b_l^m$, $c_l^n$, $d_l^m$, $e_l^n$, $f_l^m$, $g_l^n$, $h_l^m$, $i_l^n$ and $j_l^m$ are the corresponding supplement coefficients. The auxiliary polynomial functions $\varsigma_l^a(x)$ and $\varsigma_l^b(y)$ are expressed as follows:

$$\zeta_1^a(x) = x\left(\frac{x}{a} - 1\right)^2 \qquad \zeta_2^a(x) = \frac{x^2}{a}\left(\frac{x}{a} - 1\right) \qquad \zeta_1^b(y) = y\left(\frac{y}{b} - 1\right)^2 \qquad \zeta_2^b(y) = \frac{y^2}{b}\left(\frac{y}{b} - 1\right) \tag{24}$$

It can be verified that,

$$\zeta_1^a(0)=\zeta_1^a(a)=\zeta_1^{a'}(a)=0,\quad \zeta_1^{a'}(0)=1 \qquad \zeta_2^a(0)=\zeta_2^a(a)=\zeta_2^{a'}(0)=0,\quad \zeta_2^{a'}(a)=1$$
$$\zeta_1^b(0)=\zeta_1^b(b)=\zeta_1^{b'}(b)=0,\quad \zeta_1^{b'}(0)=1 \qquad \zeta_2^b(0)=\zeta_2^b(b)=\zeta_2^{b'}(0)=0,\quad \zeta_2^{b'}(b)=1 \tag{25}$$

Alternately, all the expansion coefficients in Equations (19)-(23) can be treated equally and independently as the generalized coordinates and solved from the Rayleigh-Ritz method. The vibration characteristic equation can be summed up in matrix form:

$$(\mathbf{K}-\omega^2\mathbf{M})\mathbf{G}=0 \tag{26}$$

where **K** and M denote the stiffness matrix and mass matrix of the plate, respectively.

## 3 NUMERICAL RESULTS AND DISCUSSION

In this part, numerical results for the free vibration of laminated plates with various geometric parameters and general conditions are presented. Firstly, the reliability and accuracy of the proposed method are validated by comparing with those results in the literature and those of Finite Element Analysis. Secondly, some numerical results for laminated plates with different geometric parameters and angle-plies are obtained for various boundary conditions. Finally, the effects of linear Winkler and Pasternak foundation parameters on frequencies are also illustrated.

A symmetrically laminated plate with completely free boundary conditions is considered to demonstrate the accuracy of modified Fourier method. The material properties and geometric parameters of the plate are given as follows: $a/b=3/2$, $h/a=0.1$, $E_1/E_2=20$, $\mu_{12}=0.25$, $G_{12}=0.5E_2$, $G_{13}=0.5E_2$, $G_{23}=0.33E_2$ and the truncation number is $M$ = $N$ = 11, 12 and 13. In Table 1, the first six frequency parameters $\Omega=\omega a^2\sqrt{\rho h/D}$ for the plate with angle-ply [30°/-30°/-30°/30°] are examined. It can be seen that the solutions are in agreement with the results obtained from FEM.

*Table 1 - Comparison of the first six frequency parameters Ω for a completely free laminated plate.*

| Mode | 11×11 | | 12×12 | | 13×13 | |
|---|---|---|---|---|---|---|
| | Present | FEM | Present | FEM | Present | FEM |
| 1 | 0.7497 | 0.7499 | 0.7140 | 0.7141 | 0.7052 | 0.7051 |
| 2 | 1.0033 | 1.0034 | 0.9563 | 0.9565 | 0.9473 | 0.9472 |
| 3 | 2.2191 | 2.2191 | 2.1951 | 2.1952 | 2.1472 | 2.1472 |
| 4 | 2.8798 | 2.8797 | 2.7302 | 2.7305 | 2.6934 | 2.6935 |
| 5 | 3.7860 | 3.7857 | 4.7628 | 3.7630 | 3.6789 | 3.6792 |
| 6 | 4.2841 | 4.2837 | 4.2405 | 4.2406 | 4.1870 | 4.1875 |

Table 2 compares the lowest five frequency parameters with CCCC and SCSC boundary conditions. The geometric and materials properties of the plate are: $a/b=1/2$, $E_1/E_2=20$, $E_2=E_3=10\text{GPa}$, $\mu_{12}=\mu_{13}=0.25$, $G_{12}=G_{13}=G_{23}=5\text{GPa}$ $\mu_{23}=0.3$. Three different thickness-length ratios, $h/a$ = 0.1, 0.2 and 0.3, corresponding to the laminated plates, are considered. The solutions by the three-dimensional elasticity method by Jin et al.[15] are provided for the comparisons. The maximum discrepancy does not exceed 0.048%, which is acceptable. It can be seen that the augmentation of the thickness-length ratio results in the decrease of the frequency parameters.

*Table 2 - The lowest five frequency parameters Ω for laminated plate with different conditions and thickness-leigh ratios.*

| $h/a$ | Method | Mode Number | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 |
| CCCC boundary condition | | | | | | |
| 0.1 | Present | 12.786 | 13.251 | 14.464 | 16.687 | 19.954 |
| | Ref. 15 | 12.767 | 13.243 | 14.451 | 16.647 | 19.938 |
| 0.2 | Present | 7.5301 | 8.0875 | 9.3814 | 10.203 | 11.426 |
| | Ref. 15 | 7.5325 | 8.0882 | 9.3822 | 10.210 | 11.435 |
| 0.3 | Present | 5.2978 | 5.8816 | 6.8091 | 7.0854 | 8.7982 |
| | Ref. 15 | 5.2982 | 5.8807 | 6.8086 | 7.0848 | 8.7975 |
| SCSC boundary condition | | | | | | |
| 0.1 | Present | 8.3205 | 8.3316 | 9.1983 | 11.075 | 14.015 |
| | Ref. 15 | 8.3199 | 8.3304 | 9.1970 | 11.061 | 13.998 |
| 0.2 | Present | 4.1647 | 6.1715 | 6.9588 | 8.3296 | 8.5426 |
| | Ref. 15 | 4.1652 | 6.1728 | 6.9602 | 8.3314 | 8.5467 |
| 0.3 | Present | 2.7767 | 4.7178 | 5.4358 | 5.5530 | 6.7761 |
| | Ref. 15 | 2.7768 | 4.7187 | 5.4366 | 5.5551 | 6.7789 |

Numerous new results of frequencies parameters are presented in Tables 3 and 4 for laminated plates with different boundary conditions. In the case of Table 3, the geometrical parameters are taken to be $h/a = 0.1$, $a/b = 0.5$ with various anisotropic degrees [0/90°], [0°/90°/0] and [0/90°/90°/0]. And the first two frequency parameters with three classical boundary conditions and four different anisotropic degrees, i.e. $E_1/E_2 = 5$, 10, 20, 50 are listed, respectively. The frequencies of the plates increase as the anisotropic ratio increases.

*Table 3 - The first two frequency parameters Ω for [0/90°], [0°/90°/0] and [0/90°/90°/0] laminated plates with different anisotropic degrees.*

| Boundary | $E_1/E_2$ | [0/90°] | | [0°/90°/0] | | [0/90°/90°/0] | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 1 | 2 | 1 | 2 |
| FFFF | 5 | 1.9541 | 2.1457 | 1.6814 | 2.1398 | 2.1548 | 3.4791 |
| | 10 | 2.1894 | 2.3145 | 1.7365 | 2.1448 | 2.3459 | 3.6463 |
| | 20 | 2.4016 | 3.0793 | 2.0149 | 2.1674 | 2.6447 | 3.9214 |
| | 50 | 2.4398 | 3.8621 | 2.2841 | 2.6082 | 3.0199 | 4.2553 |
| SSSS | 5 | 5.1410 | 7.7645 | 6.3545 | 8.3443 | 7.0141 | 8.9712 |
| | 10 | 6.3544 | 9.0048 | 8.7534 | 10.003 | 9.3645 | 11.045 |
| | 20 | 8.1428 | 10.217 | 11.978 | 13.014 | 12.258 | 14.247 |
| | 50 | 10.769 | 12.544 | 15.871 | 16.978 | 16.247 | 17.688 |
| CCCC | 5 | 8.2161 | 11.341 | 12.573 | 14.211 | 13.547 | 16.544 |
| | 10 | 10.357 | 12.698 | 16.144 | 17.599 | 18.691 | 21.019 |
| | 20 | 12.105 | 15.017 | 21.028 | 23.104 | 23.100 | 26.211 |
| | 50 | 15.764 | 18.350 | 27.984 | 29.012 | 30.215 | 32.887 |

In the case of Table 4, the lowest four frequencies for four-layered [45°/-45°/45°/-45°] laminated plates with seven types of boundaries (FFFF, FCFC, FSFS, SSSS, FCCC, SCSC and CCCC) are presented. The aspect ratio is chosen to be $a/b = 1.2$, and thickness-length ratio $h/a = 0.2$ is used in the calculation. The materials parameters are $E_1$=$10E_2$, $E_2 = 2\text{GPa}$, $\mu_{12} = 0.25$, $G_{12} = 0.5E_2$, $G_{13} = 0.5E_2$. The frequencies of the laminated plates with SSSS, FCCC, SCSC and CCCC are higher than other boundaries, this is because the larger restraints at the edges increase the flexural rigidity of the plates and lead to higher vibration response. Meanwhile, the corresponding lowest six mode shapes in Table 4 that laminated plates with CCCC and SCSC boundary conditions are depicted in Figs 2 and 3, respectively. Next, the effects of linear Winkler and Pasternak foundation parameters on frequencies parameters are illustrated.

*Table 4 - The lowest four frequency parameters Ω for four-layer, angle-ply [45°/-45°/45°/-45°] with different boundaries.*

| Boundary conditions | Mode 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| FFFF | 2.2375 | 5.6794 | 7.3487 | 10.985 |
| FCFC | 3.0076 | 6.8245 | 8.1568 | 14.763 |
| FSFS | 1.1275 | 5.0318 | 5.8762 | 11.541 |
| SSSS | 8.8766 | 14.847 | 21.674 | 24.698 |
| FCCC | 5.0348 | 10.593 | 15.975 | 17.234 |
| SCSC | 9.5481 | 15.157 | 24.758 | 25.084 |
| CCCC | 14.392 | 20.381 | 28.537 | 36.432 |

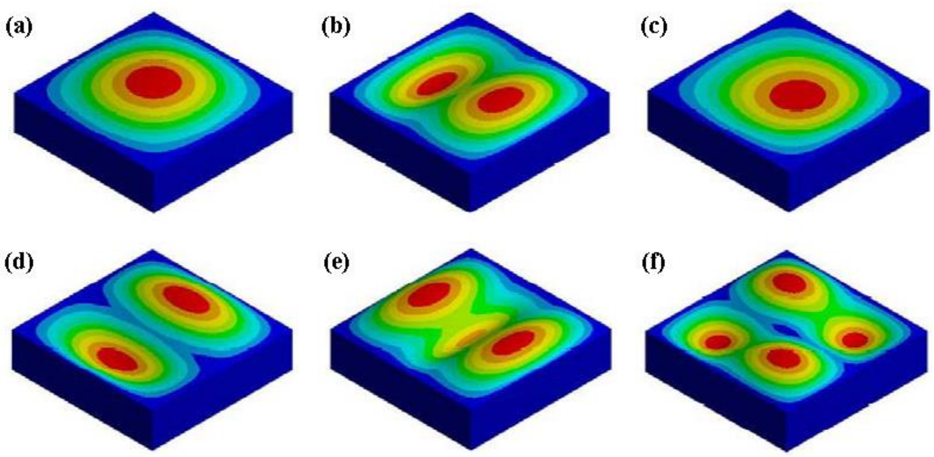


*Fig - 2. The first six mode shapes of the laminated composite plates with CCCC boundary condition: (a) m=1; (b) m=2; (c) m=3; (d) m=4; (e) m=5; (f) m=6*

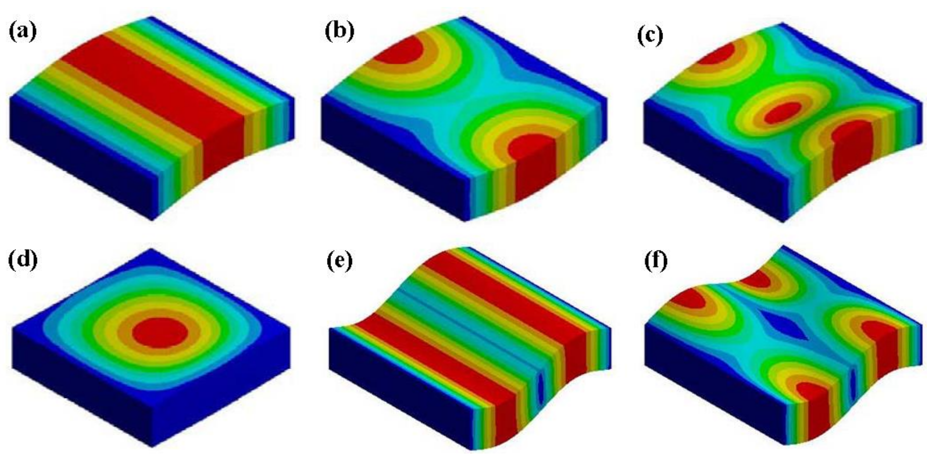


*Fig - 3. The first six mode shapes of the laminated composite plates with SCSC boundary condition: (a) m=1; (b) m=2; (c) m=3; (d) m=4; (e) m=5; (f) m=6*

The variation of the frequencies parameters $\Delta\Omega$ through the linear Winkler and Pasternak foundation parameters ($k_w$ and $k_s$) is shown in Fig 4. It is obvious that the change in frequencies parameters is very small when $k_w$ and $k_s$ is less than $10^5$, while when the values of $k_w$ and $k_s$ are between $10^5$ and $10^9$, the frequencies parameters increase rapidly, which is a sensitive range. In the case of greater than $10^9$, although the stiffness of $k_w$ and $k_s$ has a wide range of change, frequencies parameters rarely change.

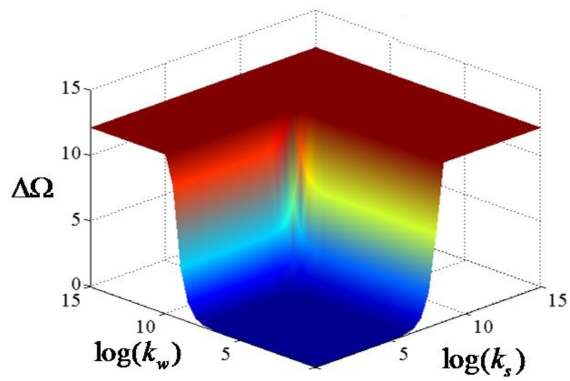


*Fig - 4. The variation of the frequencies parameters through Winkler and Pasternak foundation parameters.*

## 4 CONCLUSIONS

In this paper, free vibration of laminated composite rectangular plates with general boundary conditions has been performed with a modified Fourier series method. The displacement and rotation in each direction for laminated plates is described as an improved form of double Fourier cosine series and several closed-form auxiliary functions to eliminate any possible jumps and boundary discontinuities. Exact solutions are obtained by Rayleigh-Ritz method. And the comparisons between the results show that the presented method has high accuracy and reliability.

The effects of boundary conditions, geometric parameters and linear Winkler and Pasternak foundation parameters on the frequencies are illustrated comprehensively. Finally, numerous new results for laminated composite rectangular plates with different geometric parameters and angle-plies are also presented for future research.

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