Free vibration analysis of rectangular thin plate with multiple openings under general boundary conditions

Rui Nie [a]
Tianyun Li [b]
Xiang Zhu [c]
Wenjie Guo [d]
[1]School of Naval Architecture and Ocean Engineering, Huazhong University of Science and Technology, Wuhan 430074, China
[2]Collaborative Innovation Center for Advanced Ship and Deep-sea Exploration, Shanghai 200240, China
[3]Hubei Key Laboratory of Naval Architecture and Ocean Engineering Hydrodynamics,Wuhan 430074, China

ABSTRACT

The free vibration characteristics of a rectangular thin plate with series of openings are studied based on the Rayleigh-Ritz method in this paper. Firstly, the strain energy and kinetic energy of the plate are calculated utilizing the modified Fourier series. Then, uniformly distributed transitional and rotational springs are applied to deal with general boundary supports, and the elastic potential energy of the springs can be obtained. Furthermore, the plate is divided into several parts according to its amounts of openings and the energy of each parts are calculated separately and the spring stiffness of cut line between two separated parts are the same. Finally, the governing equation of the plate is obtained with the energy functional variation method. The present method is proved to be accurate by comparing the natural frequencies with those calculated by the finite element method. Besides, the influence of the amounts of openings and the area of total openings are discussed.

1 INTRODUCTION

Plates are structural elements ordinarily used in structures in almost all branches of engineering, naval architecture, ocean engineering and so on. The one of most important engineering problems accompanied with the plates can be classified into one main group:

[a] email: nierui2016@hust.edu.cn
[b] email: ltyz801@hust.edu.cn
[c] email: zhuxiang@hust.edu.cn
[d] email: 739633869@qq.com

vibration. If the plates are not work under water, they are often arranged with different types of openings for many reasons, such as weight reduction, altering the natural frequencies, outfitting and providing accessibility to other parts of the structure[1,2]. Besides, the usual ship deck structure can be regarded as plating with different openings. In the actual process, these structures suffer excessive vibration.

Literature reviews on the vibration analysis of rectangular plates with openings are available in Cho et al.[2] and Kwak and Han[3] where different approaches are discussed. Experimentally obtained natural frequencies of rectangular plates with elliptical inner boundaries are reported by Nagaya[4]. Hegarty and Ariman[5] investigated the free vibrations of rectangular plates with circular openings by a least-squares point-matching method. Different variants of the Rayleigh–Ritz method are considered by Ali and Atwal[6] Lam et al.[7]. Lee et al.[1] and Avalos and Laura[8]. In Chang and Chiang[9] Hamilton's variational principle and potential theory are used to derive the governing equations for a Mindlin plate and quadratic element of eight nodes is used. Recently, Huang[10] has investigated the effects of constraints, the circular opening and in-plane loading on the vibration of rectangular plates. To study the interactions between the plate and the hole, he adopted the so-called receptance approach. While a receptance reduction is used at the plate–hole interface, a receptance addition is used for the plate stiffeners at the clamped edge. In addition, Chen et al.[11] analyzed the flexural and in-plane vibration of elastically restrained thin rectangular plates with openings using the Chebyshev series and Lagrangian method. By setting groups of boundary springs and assigning corresponding stiffness constants to the springs, the general boundary conditions, including all the classical boundary conditions, are achieved.

At present, the FEM (finite element method) is probably the most important tool for the vibration problem of perforated plates. However, there are still some shortcomings in its application, such as the time limited model preparation, so that the FEM is only applicable to the final structural inspection, and all the dimensions and boundary conditions are known. However, at the early stage of design, when the size of the plate, the edge constraints, and the shape, size, and layout of the opening are influenced by the dynamic response of the plate, some simple methods can be useful.

A simplified energy-based method for the natural vibration analysis of rectangular Mindlin plates with openings and arbitrary boundary conditions is presented by Cho et al.[2]. The procedure is based on the assumed mode method (AMM)[12-14] and it can be applied to plates with multiple free-edge openings, arbitrarily placed within the plate area. This concept is actually very similar to the Rayleigh–Ritz method, but here the eigenvalue problem is derived by applying Lagrange's equation instead of minimizing the energy functional. However, the potential and kinetic energies of the system should still be expressed in a convenient manner. The opening is accounted for by taking the potential and kinetic energy of the cut-out parts away from the corresponding total plate energies without openings.

In this article, the AMM is applied for the natural vibration analysis of rectangular thin plates with a different number and size of circular openings. In order to illustrate the application of the AMM and investigate its accuracy, clamped plates with free circular inner openings are analyzed. For rectangular plates with different number of openings, an FEM analysis is performed to evaluate the approximate AMM results. The reason is that in this case the FEM solution for the plate can be determined as a referent one. The cases of different sizes of openings and the natural frequency of the plate under the corresponding condition are discussed, also.

2 FORMULA

The plate model considered in this study is a rectangular plate with multiple openings, and general boundary constraints are applied on the rectangle plate's outside edges and the central opening's edges. As shown in Fig.1, the whole plate is divided into several same plates and a circular hole is cut out from the central area of each small plate. The same plates can be numbered from 1,2 to I in turn. The length and breadth of the small plate is X and Y. Thus, the total length of the whole plate in the y direction is IY. The circular hole's radius is R . The part of the opening's area is S_0 and the solid part of the small plate's area is S.

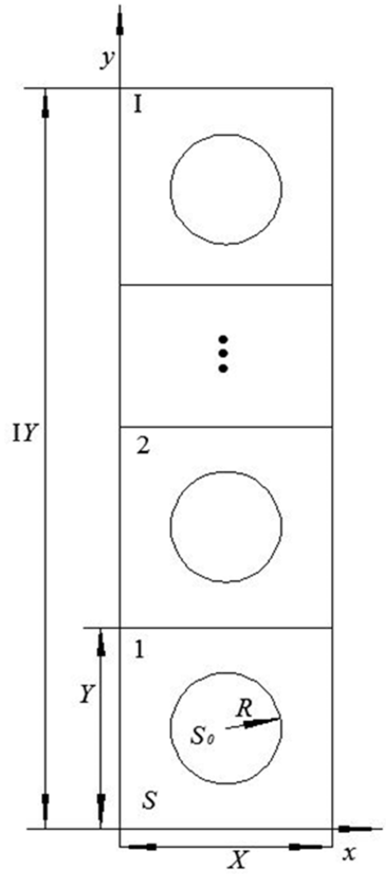

Fig. 1 – Rectangular plate with multiple openings.

For each small plate, the strain energies and kinetic energies of the plate, $V_p^{[i]}$ and $T^{[i]}$, is written as

$$V_p^{[i]}=\frac{1}{2}D\iint_S\left(\frac{\partial^2 w^{[i]}(x,y)}{\partial x^2}\right)^2+\left(\frac{\partial^2 w^{[i]}(x,y)}{\partial y^2}\right)^2 +2\mu\left(\frac{\partial^2 w^{[i]}(x,y)}{\partial x^2}\right)\left(\frac{\partial^2 w^{[i]}(x,y)}{\partial y^2}\right) +2(1-\mu)\left(\frac{\partial^2 w^{[i]}(x,y)}{\partial x\partial y}\right)^2 \mathrm{d}x\mathrm{d}y \tag{1}$$

$$T^{[i]}=\frac{1}{2}\rho h\omega^2\iint_S w^{[i]}(x,y)^2\,\mathrm{d}x\mathrm{d}y \tag{2}$$

Where $w^{[i]}(x,y)$ is the flexural displacement function of each small thin plate, i is the number of the plate (i=1,2,..I) and the time term e^{jwt} is omitted for brief in this study, where w is the circular frequency. h is plate's thickness, r is material density, m is Poisson's ratio, $D=Eh^3/(12(1-\mu^2))$ is the plate bending stiffness and E is elastic modulus.

The actually strain energies and kinetic energies of the plate can be regarded as the energies of a plate without hole minus the energies of the opening. Such as

$$\begin{aligned} T^{[i]} &= \frac{1}{2}\rho h\omega^2 \iint_S w^{[i]}(x,y)^2 \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\rho h\omega^2 \int_0^X \int_0^Y w^{[i]}(x,y)^2 \mathrm{d}x\mathrm{d}y - \frac{1}{2}\rho h\omega^2 \iint_{S_0} w^{[i]}(x,y)^2 \mathrm{d}x\mathrm{d}y \end{aligned} \tag{3}$$

The expression of the actually strain energies can also be written in a similar form.

The permissible flexural displacement function $w^{[i]}(x,y)$ can be written as

$$w^{[i]}(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn} f_m(x) g_n(y) \tag{4}$$

where $f_m(x)$ and $g_n(y)$ are the orthogonal polynomials with respect to x and y, satisfying the specified elastic edge constraints. A_{mn} are the unknown influence coefficients of orthogonal polynomials, m and n are sequence numbers of polynomials, M and N are the truncated numbers of orthogonal polynomials, respectively.

The modified Fourier series are adopted to substitute for $f_m(x)$ and $g_n(y)$, which can be written as

$$\begin{cases} f_m(x) = \sin(m\pi \dfrac{x}{X}) & 0 < m < 5 \\ f_m(x) = \cos(m\pi \dfrac{x}{X}) & m \geq 5 \end{cases} \qquad m = 1,2,3...M \tag{5a}$$

$$\begin{cases} g_n(y) = \sin(n\pi \dfrac{y}{Y}) & 0 < n < 5 \\ g_n(y) = \cos(n\pi \dfrac{y}{Y}) & n \geq 5 \end{cases} \qquad n = 1,2,3...N \tag{5b}$$

3 THE ELASTIC POTENTIAL ENERGY OF THE EDGES

The boundary condition of the plate can be regarded as homogeneously distributed spring mounted on the plate's edges whose direction is perpendicular to the plate area, including the translational springs and the rotational springs, as shown in Fig.2. k and K are the stiffness of the translational spring and rotational spring at the plate's boundary, respectively.

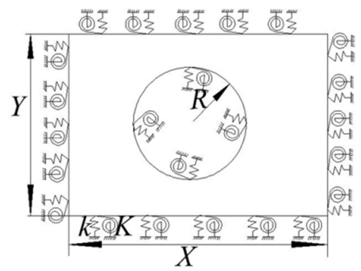

Fig. 2 – Rectangular plate with central circular opening supported by springs.

The stiffness k and K can be different according to various types of boundary conditions. Three kinds of classical boundary conditions and the corresponding stiffness of the virtual spring are listed in Tab.1.

Table 1 - Stiffness settings of three different boundary conditions

	Clamped	Simply supported	Free
k（N/m）	∞	∞	0
K(N·m /rad)	∞	0	0

Therefore, the elastic potential energy that the springs on the straight edges store can be written as

$$
\begin{aligned}
V_s^{[1]} &= \frac{1}{2}\int_0^X \left\{ \begin{aligned} &[k_{y0} w^{[1]}(x,y)^2 + K_{y0}(\frac{\partial w^{[1]}(x,y)}{\partial y})^2]_{y=0} \\ &+[k_{yY}(w^{[1]}(x,y) - w^{[2]}(x,y))^2 + K_{yY}(\frac{\partial w^{[1]}(x,y)}{\partial y} - \frac{\partial w^{[2]}(x,y)}{\partial y})^2]_{y=Y} \end{aligned} \right\} \mathrm{d}x \\
&+ \frac{1}{2}\int_0^Y \left\{ \begin{aligned} &[k_{x0} w(x,y)^2 + K_{x0}(\frac{\partial w(x,y)}{\partial x})^2]_{x=0} \\ &+[k_{xX} w(x,y)^2 + K_{xX}(\frac{\partial w(x,y)}{\partial y})^2]_{x=X} \end{aligned} \right\} \mathrm{d}y \\
V_s^{[2]} &= \frac{1}{2}\int_0^X \left\{ \begin{aligned} &[k_{yY}(w^{[2]}(x,y) - w^{[1]}(x,y))^2 + K_{yY}(\frac{\partial w^{[2]}(x,y)}{\partial y} - \frac{\partial w^{[1]}(x,y)}{\partial y})^2]_{y=0} \\ &+[k_{yY}(w^{[2]}(x,y) - w^{[3]}(x,y))^2 + K_{yY}(\frac{\partial w^{[2]}(x,y)}{\partial y} - \frac{\partial w^{[3]}(x,y)}{\partial y})^2]_{y=Y} \end{aligned} \right\} \mathrm{d}x \\
&+ \frac{1}{2}\int_0^Y \left\{ \begin{aligned} &[k_{x0} w(x,y)^2 + K_{x0}(\frac{\partial w(x,y)}{\partial x})^2]_{x=0} \\ &+[k_{xX} w(x,y)^2 + K_{xX}(\frac{\partial w(x,y)}{\partial y})^2]_{x=X} \end{aligned} \right\} \mathrm{d}y \\
&\cdots \\
V_s^{[I]} &= \frac{1}{2}\int_0^X \left\{ \begin{aligned} &[k_{yY}(w^{[I]}(x,y) - w^{[I-1]}(x,y))^2 + K_{yY}(\frac{\partial w^{[I]}(x,y)}{\partial y} - \frac{\partial w^{[I-1]}(x,y)}{\partial y})^2]_{y=0} \\ &+[k_{yY} w^{[I]}(x,y)^2 + K_{yY}(\frac{\partial w^{[I]}(x,y)}{\partial y})^2]_{y=Y} \end{aligned} \right\} \mathrm{d}x \\
&+ \frac{1}{2}\int_0^Y \left\{ \begin{aligned} &[k_{x0} w(x,y)^2 + K_{x0}(\frac{\partial w(x,y)}{\partial x})^2]_{x=0} \\ &+[k_{xX} w(x,y)^2 + K_{xX}(\frac{\partial w(x,y)}{\partial y})^2]_{x=X} \end{aligned} \right\} \mathrm{d}y
\end{aligned} \tag{6}
$$

where k_{x0} , K_{x0} , k_{y0} , K_{y0} , k_{xX} , K_{xX} , k_{yY} and K_{yY} are the stiffness of the translational and rotational springs of four external edges of the small rectangular plate respectively, as Fig. 3 shows.

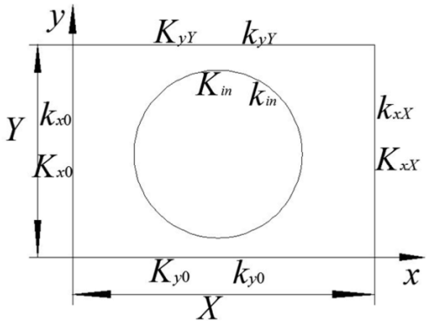

Fig. 3 – Stiffness of the translational and rotational springs on all edges of the plate.

If the curve edge of the opening is also constrained with elastic support, and the boundary condition is regarded as free (k_{in}= K_{in}=0), the elastic potential energy that the springs of the opening's curve edges store, always be equal to 0.

From the above derivation, the Lagrange energy functional of the system can be written as

$$\Pi = \sum V_p^{[i]} + \sum V_s^{[i]} - \sum T^{[i]} - W_e \tag{7}$$

where W_e is the work done by external periodical force and W_e=0 for the free vibration. By substituting Eq. (1), (2) and (6) into the Lagrange's equation of motion below

$$\frac{\partial \Pi}{\partial A_{mn}} = 0 \tag{8}$$

For free vibration analysis, the discrete matrix equation can be obtained as

$$([\boldsymbol{K}] - \omega^2[\boldsymbol{M}])\{\boldsymbol{A}\} = \{\boldsymbol{0}\} \tag{9}$$

4 NUMERICAL EXAMPLES

4.1 The Convergence Analysis

From Eq. (4), there exists the summation of series in the expression, therefore the accuracy of the developed procedure is seriously affected by the chosen value of M and N. Then, from Eq. (6), the elastic potential energy that the springs of the edges store is seriously affected by the chosen value of k and K.

In order to demonstrate the astringency of the developed procedure, a rectangular plate with three central circular openings is analyzed. As it is shown in Fig. 1, the length and breadth of the small plates are X and Y, and the length of the radius is equal to R, respectively. The study is carried out for the cases X=6m, Y=6m, and R=1m. The Young's modulus E, Poisson's ratio m, material density ρ and plate thickness h are set to 2.1×10^{11} N/m^2, 0.3, 7850kg/m^3 and 0.02m, respectively, which will be also used in the following examples. The model is clamped at outlines and free at the opening edge (simplified as C-C-C-C-F).

First of all, the stiffness of the translational and rotational springs are set as a suitable large constant value 1×10^{14} N/m to model the clamped boundary condition, while M and N are set from 8 to 16 to examine the convergence of the numerical calculation. The natural frequencies obtained with different values of M and N are listed in Tab. 2. It can be found that the natural frequencies of the first ten orders become stable for the case of M=N=14, therefore M and N will also be set as 14 in the following calculations.

Table 2 - Natural frequencies of multiple openings plate with three central circular openings (Hz), C-C-C-C-F

Modes order	M=N=8	M=N=10	M=N=12	M=N=14	M=N=16	M=N=18
1	3.37	3.34	3.32	3.31	3.31	3.31
2	3.71	3.68	3.65	3.63	3.63	3.62
3	4.78	4.68	4.64	4.62	4.61	4.61
4	5.76	5.65	5.59	5.54	5.52	5.51

5	7.31	7.17	7.10	7.03	7.02	7.00
6	8.58	8.37	8.17	7.98	7.92	7.89
7	8.69	8.55	8.44	8.29	8.25	8.22
8	9.03	8.91	8.80	8.66	8.62	8.58
9	9.88	9.61	9.37	9.10	9.03	8.98
10	10.72	10.49	10.34	10.14	10.08	10.04

The variety of the natural frequency with K and k for M=N=14 is investigated according to the developed procedure, and in Tab.3, natural frequency become uniform from K=k=1×10^6 to larger, therefore, the method become convergent for K=k=1×10^{12}, and it also can be adopted into posterior calculation.

Table 3 - Natural frequencies of multiple openings plate with different value of k and K (Hz), C-C-C-C-F

Modes order	K=k=10^6	K=k=10^8	K=k=10^{10}	K=k=10^{12}	K=k=10^{14}
1	2.86	3.31	3.31	3.31	3.31
2	3.14	3.62	3.63	3.63	3.63
3	3.92	4.60	4.61	4.61	4.61
4	4.69	5.51	5.52	5.52	5.52
5	5.84	7.00	7.02	7.02	7.02
6	6.59	7.90	7.92	7.92	7.92
7	6.71	8.23	8.25	8.25	8.25
8	6.84	8.59	8.62	8.62	8.62
9	7.16	9.00	9.03	9.03	9.03
10	7.92	10.06	10.08	10.08	10.08

4.2 The Accuracy Analysis

Taking the amounts of openings into consider, the whole rectangular plate will be divided into different blocks according to the amounts, and the total area of the openings will also remain unchanged.

The obtained results are compared with the results obtained by the FEM (ANSYS), and the comparisons of the natural frequencies are given in Tab.4 to Tab.6, in which the former ten natural frequencies are displayed.

The error between two different methods can be written as

$$error = \left| \frac{f_F - f_P}{f_F} \right| \times 100\% \tag{10}$$

where f_F is the numerical results obtained by the FEM software ANSYS and f_P is the solution by the present method.

Table 4 - Natural frequencies of rectangular plate with two openings (Hz), C-C-C-C-F

Modes order	f_F	f_P	Errors(%)
1	3.18	3.20	0.78
2	3.30	3.34	1.17
3	3.99	4.03	1.06
4	4.64	4.79	3.16
5	5.05	5.25	3.96
6	6.42	6.67	3.94
7	8.29	8.21	0.92
8	8.46	8.42	0.51
9	9.17	8.94	2.42
10	9.35	9.27	0.88

Table 5 - Natural frequencies of rectangular plate with three openings (Hz), C-C-C-C-F

Modes order	f_F	f_P	Errors(%)
1	3.32	3.31	0.57
2	3.63	3.63	0.53
3	4.62	4.61	0.45
4	5.52	5.52	0.33
5	7.02	7.02	0.39
6	7.91	7.92	1.12
7	8.26	8.25	1.13
8	8.63	8.62	1.00
9	9.02	9.03	0.85
10	10.07	10.08	1.41

Table 6 - Natural frequencies of rectangular plate with four openings (Hz), C-C-C-C-F

Modes order	f_F	f_p	Errors(%)
1	3.30	3.23	2.03
2	3.65	3.60	1.44
3	4.25	4.35	2.19
4	5.55	5.63	1.44
5	6.80	6.78	0.35
6	8.29	8.11	2.23
7	8.45	8.24	2.55
8	8.65	8.62	0.29
9	9.28	9.46	1.95
10	10.00	10.50	4.94

As the tables show (Tab. 4 to Tab. 6), all the relative errors are less than 5%, which accounts for the accuracy of the present method. And, limited to time, for the lack of reference samples, there seems to be no obvious rule of the natural frequencies' variety between the rectangular plate with different amounts of openings.

4.3 Natural Frequencies of Rectangular Plate with Different Opening Area

Since the accuracy of the method has been proved, this method is used to study the relationship between the area of openings and the natural frequency of the plate, and the amounts of openings remain unchanged. The object of the study is the three openings plate in the above, except for the opening's radius, the other parameters are constants. The first five frequencies are presented in Tab. 7, the opening's radius R was varied for values of 0.5 to 2.5 m.

Table 7 - Natural frequencies of multiple openings plate with different open area (Hz), C-C-C-C-F

Modes order	R=0.5	R=1.0	R=1.5	R=2.0	R=2.5
1	3.21	3.31	3.51	3.70	4.02
2	3.58	3.63	3.65	3.72	4.02
3	4.27	4.61	5.65	7.65	9.02
4	5.28	5.52	6.26	7.79	9.02
5	6.63	7.02	8.28	8.74	12.25

In order to get the free vibration characteristics of the rectangular plate with multiple openings more intuitively, the calculated data had been drawn as follows

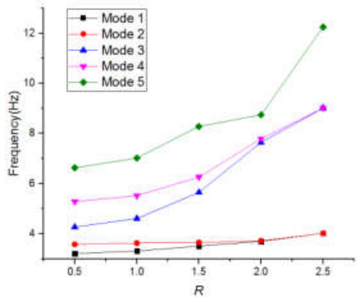

Fig. 4 – Natural frequencies of multiple openings plate with different open area (Hz), C-C-C-C-F

From Fig.4 it is seen that, for the multiple openings plate, the effect of the openings is to increase the frequency and the increase is monotonic. For curved plates, there is an increased in the frequency value when the opening size is increased. This phenomenon should be expected because when the openings' boundary is free, only the strain energies and kinetic energies of the plate decrease. Besides, the external boundaries are clamped, the elastic potential energy will be a very large value and it is invariant. From Eq. (7) and Eq. (9) , it can be simply considered that potential energy, including structure potential energy and boundary energy, determines the structure stiffness matrix value and the kinetic energy determines the structure mass matrix value. Therefore, it can be thought that the values of [$\boldsymbol{K}$] are basically maintained and the values of [$\boldsymbol{M}$] are reduced, the frequencies of multiple openings plate will be higher.

5 CONCLUSIONS

The free vibration problems are disposed through eigenvalue decomposition of stiffness matrix obtained based on the Rayleigh-Ritz method. The modified Fourier series is adopted to substitute the shape function of the thin rectangular plate. During the study, general boundary supports are imposed on the opening's edges and rectangular plates' straight sides and simulated by uniformly distributed transitional and rotational springs.to achieve the elastic potential energy of arbitrary boundary conditions. After that, the astringency of the developed procedure are well testified by analyzing the vibration characteristics of the rectangular plate with three central circular shaped openings. In addition, it should be mentioned that the selected value of the spring stiffness k and K as well as the number of orthogonal polynomials M and N should be large enough to make the calculation more precise, but it should not be too large to consume too much computational time. What's more, three different amounts of the openings are applied on the rectangular plate served as numerical example. Natural frequencies of result are compared with the obtained result of FEM, which reveals that they come close to accordance.

Theoretically, the method can be applied on the arbitrarily-shaped thin plate with classical boundary conditions and multiple arbitrarily-shaped openings in addition to the rectangular plate analyzed above, thus further study for the vibration characteristics of those plate should be carried on. It should be mentioned that the position of the opening in the plate area can be arbitrary and the calculation might be simplified for some symmetrical plate as plates' symmetry axis lines can also be transformed into spring-connected edges.

6 ACKNOWLEDGEMENTS

Authors wish to express their gratitude to the National Natural Science Foundation of China (Contract No. 51579109 and 51479079) and the Chinese Fundamental Research Funds for the Central Universities (HUST: 2016YXZD010) that have supported this work.

7 REFERENCES

1. Lee, H.P., Lim, S.P., Chow, S.T., 1990, "Prediction of natural frequencies of rectangular plates with rectangular cutouts," *Comput Struct*, *Vol. 36*, pp. 861–869.
2. Cho, D.S., Vladimir, N., Choi, T.M., "Approximate natural vibration analysis of rectangular plates with openings using assumed mode method," *Int J Nav Archit Ocean Eng 2013*, *Vol 5*, pp. 478–491.
3. Kwak, M.K., Han, S., "Free vibration analysis of rectangular plate with a hole by means of independent coordinate coupling metho," *J Sound Vib 2007*, *Vol 306*, pp. 12–30.
4. Nagaya, K., "Simplified method for solving problems of vibrating plates of doubly connected arbitrary shape, part II: applications and experiments," *J Sound Vib 1981*, *Vol 74*, pp. 553–564.
5. Hegarty, R.F., Ariman, T., "Elasto- dynamic analysis of rectangular plates with circular holes," *Int J Solids Struct 1975*, *Vol 11*, 895–906.
6. Ali, R., Atwal, S.J., "Prediction of natural frequencies of vibration of rectangular plates with rectangular cutouts," *Comput Struct 1980*, *Vol 12*, pp. 819–823.
7. Lam, K.Y., Hung, K.C., Chow, S.T., "Vibration analysis of plates with cutouts by the modified Rayleigh-Ritz method," *Appl Acoust 1989*, *Vol 28*, pp. 49–60.
8. Avalos, D.R., Laura, PAA., "Transverse vibrations of simply supported rectangular plates with two rectangular cutouts," *J Sound Vib 2003*, *Vol 267*, pp. 967–977.
9. Chang, C.N., Chiang, F.K., "Vibration analysis of a thick plate with an interior cut-out by a finite element method," *J Sound Vib 1988*, *Vol 125*, pp. 477–486.
10. Huang, D.T., "Effects of constraint, circular cutout and in-plane loading on vibration of rectangular plates," *Int J Mech Sci*, *Vol 68*, pp. 114–124.
11. Chen, Y., Jin, G., Liu, Z., "Flexural and in-plane vibration analysis of elastically restrained thin rectangular plate with cutout using Chebyshev–Lagrangian method," *Int J Mech Sci 2014*, *Vol 89*, pp. 264–278.
12. Chung, J.H., Chung, T.Y., Kim, K.C., "Vibration analysis of orthotropic Mindlin plates with edges elastically restrained against rotation," *J Sound Vib 1993*, *Vol 163*, pp. 151–163.
13. Kim, K., Kim, B.H., Choi, T.M., "Free vibration analysis of rectangular plate with arbitrary edge constraints using characteristic orthogonal polynomials in assumed mode method," *Int. J. Nav. Arc. Oce. Eng. 2012*, *Vol 4*, pp. 267–280.
14. Cho, D.S., Vladimir, N., Choi, T.M., "Naturalvibration analysis of stiffened panels with arbitrary edge constraints using the assumed mode method. Proc IMechE, Part M: J Engineering for the Maritime Environment," Epub ahead of print 11 February 2014. DOI: 10.1177/1475090214521179.